Scientific African 10 (2020) e00643

Contents lists available at ScienceDirect

Scientific African

journal homepage: www.elsevier.com/locate/sciaf

Some characterizations of equivalence relation on contraction mappings

S.A. Akinwunmi[a,*], M.M. Mogbonju[b], G.R. Ibrahim[c]

[a] Department of Mathematics and Computer Science, Faculty of Science, Federal University of Kashere, P.M.B, 0182, Gombe, Nigeria
[b] Department of Mathematics, Faculty of Physical Science, University of Abuja, P.M.B, 117, Abuja, Nigeria
[c] Department of Statistics and Mathematical Science, Faculty of Science, Kwara State University Malete, P.M.B, 1530, Kwara, Nigeria

ARTICLE INFO

Article history:
Received 9 February 2020
Revised 2 June 2020
Accepted 20 November 2020

mathematics subject classification:
16W22
10X21
06f05

Keywords:
Contraction mapping
Collapse
Height
Waist
Equivalence relation

ABSTRACT

Let T_n be the set of full transformations and P_n be the set of partial transformations. It is shown that T_n form a semi-group of order n^n and P_n form a semi-group of order $(n+1)^n$. Let $\rho(n,m)$ be a binary relation then we define the image set of ρ (n,m), $\mathcal{I}(\rho)$: $\{n|n \in N$ *and there exists* $m \in M : (m,n) \in \rho\}$ whenever $(\equiv \rho)$: $\equiv \rho$ on a set M is called an equivalence relation if $\equiv \rho$ is reflexive, symmetric and transitive. Then, For all $m \in M$, we let $[m]$ equivalence class denote the set $[m] = \{n \in M | n \equiv \rho m\}$ with respect to $\equiv \rho$ determine by m. Furthermore, we show that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \upsilon \mathcal{R}$ implies $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Therefore, $\mathcal{D}$ is the minimum equivalence relation class containing $\mathcal{L}$ and $\mathcal{R}$. Hence, $\mathcal{D} \subseteq \mathcal{J}$. If $n \in X_m : \{n \in X_m | n\ x\ n = n; nx = xn\}$ then $n \in \mathcal{D}_{class}$ is regular. We also show that for $\mathcal{L}_{class}$, $\mathcal{R}_{class}$ and $\mathcal{H}_{class}$ for all $m, n \in D(\alpha)$ we have α such that $D(\alpha) \subseteq M$ implies $I(\alpha) \subseteq M$. Then for any transformation of a finite semi-group $\beta, \alpha \in S$ where $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, α and β represent the five equivalence relations and CP_n represent the sub-semi-group of partial contraction mapping on $M = \{1, 2, 3 \ldots\}$ while $|Q|$ denotes the order of Q.

© 2020 The Authors. Published by Elsevier B.V. on behalf of African Institute of Mathematical Sciences / Next Einstein Initiative.
This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

Introductory ideas

If a finite transformation Semi-group S is defined by means of certain properties there is a problem of determining the elements of S explicitly. In the current manuscript, we considered the CP_n semi-group of partial contraction transformation of a set M. Our aim in this section is to introduce the basic knowledge required towards the understanding the contraction mapping and the five equivalences class in Mathematical terms.

Mathematics is described as the study of relations, since it is a set of ordered pairs, Sequence occurs naturally in the growth patterns of tree branches and many other natural structures. We observed sequence in the leaf; their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. One may

* Corresponding author.
E-mail address: sakinwunmi@fukashere.edu.ng (S.A. Akinwunmi).

https://doi.org/10.1016/j.sciaf.2020.e00643
2468-2276/© 2020 The Authors. Published by Elsevier B.V. on behalf of African Institute of Mathematical Sciences / Next Einstein Initiative. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

describe a sequence as a function (mapping) whose domain is a sub-set of counting numbers. Each number in the sequence is called a term.

Like set, mapping plays a central role in mathematics. Mapping helps to study the relationship between various algebraic structures. For the basic properties of algebraic structure refer to [1]. Transformation is used instead of mapping, the later serve as another name for the former. Refer to [2, 3] for an introduction to semi-group of mappings. We denote the domain of a transformation α $Dom(\alpha)$ and image set of α $Im(\alpha)$ by $D(\alpha)$ and $I(\alpha)$ respectively as used by Malik et al. [1]. We conclude the notation P_n for the partial transformation semi-group as used by Umar [4]. The following assertion will help to understand what are the five relations as used in the current paper.

Let $M = \{(m,n)(m,n) \in \rho; m \leq n\}$, clearly, M is a sub-set of ρ X ρ and it contains exactly all ordered pairs (m,n) for which the relation $\leq$ holds. It follows if $(m,n) \notin M$ then the relation $m \leq n$ does not hold between the real numbers m and n. Thus, we can say the set M represents the relation $\leq$ in ρ. If ρ from a set M into set N is a sub-set of M X N, then if $(m,n) \in \rho$, we write $m\rho n$ or $\rho(m) = n$. If $m\rho n$, sometimes we say that m is related to n (n is a relation with m) with respect to ρ. If $M = N$, we speak of binary relation on M.Suppose ρ is a relation on M, then: ρ is reflexive if each $m \in M$; $m\rho m$ implies there exists $m, k \in \rho$ whenever $m\rho k$ and $k\rho m$ implies for every $m, k, n \in \rho$ such that $m\rho k$ and $k\rho n$ then $m\rho n$ that is equivalence relation. For example: ρ on $k = N$ X $N : (k,p), (m,n) \in M$ then $(k,p)\rho(m.n)$ if $k+n = p+m$. If $k, p \in N$ implies for all $(k,p)\rho(k,p)$, also if $(k,p)\rho(m,n)$ then $k+n = p+m$ implies $n+k = m+p$ implies $m+p = n+k$ implies $(m,n)\rho(k,p)$. Similarly, suppose $(k,p)\rho(m,n)$and $(m,n)\rho(e,f)$ implies $k+n = k+m$ and $m+f = n+e$ we have $k+n+m+f = p+m+n+e$ implies $k+f = p+e$ such that $(k,p)\rho(e,f)$.

Let $M = \{a,b,c\}$ define on $\rho \subseteq M$ X M by $\rho = \{(a,a),(b,b)(c,c),(a,b),(b,a),\ (c,a),(a,c),(b,c)\}$. Then ρ is a relation from M into M. The usefulness of an equivalence relation lies in the fact that we can get a partition of the set under consideration through it. Each component of this partition is an equivalence class and as the equivalence relation is concerned; all the elements of equivalence will be found to have same common property. Thus, it is sufficient to study such property of only one element in an equivalence class with respect to the given relation. That is any one element in an equivalence relation is sufficient to represent that equivalence class [5].

Let S be a finite semi-group. If $M, N \subset S$, we set $MN = \{mn : m \in M; n \in N\}$. A sub-set $I \subset S$ is left (right) ideal provided that for all $m \in S$ and $n \in I$ we have $mn \in I$, and ($IS \subset I$). (One-sided ideal). A sub-set $I \subset S$ is two-sided (ideal) if it is both left (right) ideal [6]. A Semi-group S with ρ is left (right) compatible with operation on S if: for all $s,t,a \in S : (s,t) \in \rho$ implies $(as,at) \in \rho$ and $((sa,ta) \in \rho)$. It is totally compatible if for all $s,t,s',t' \in S : (s,t) \in \rho$ and $(s',t') \in \rho$ implies $(ss',tt') \in \rho$.

The notion of ideals leads naturally to the consideration of certain equivalence relations on a semi-group. These equivalences have played a fundamental role in the development of semi-group theory [7]. Some classical aspects of the structure of a finite semi-group can be described in terms of the Greens relations in which all are reduce to the universal equivalence in a group theory. The above mentioned relations characterize the element of a finite semi-group in terms of principal ideal they generated. It is convenient to define the five relations over the monoid instead of working directly on the semi-group, but if a semi-group has no identity element, we adjoined an element and name it to be an identity.

$$\text{Thus}: S^1 = \begin{cases} S, & if\ S\ has\ an\ identity \\ S \cup \{1\}, & otherwise \end{cases}$$

Where the element 1 is such that for all $m \in S$ $1(m) = m = (m)1$ and $1 * 1 = 1$. In a finite semi-group S, an element $m \in S$ is called left (right) identity of S if for all $m \in S : m * e = m$ and $(e \in S : e * m = e)$. If m is both left (right) identity then it is an identity of S.

The five equivalence relation class on a finite semi-group S are given by the rule viz: for all $m, n \in S$, $m\ \mathcal{L}\ n$ if and only if $S'm$ and $S'n$ that is $mS' = S'n$, where $m, n \in S$ generate same left ideal. It is $m\ \mathcal{R}\ n$ if and only if $mS' = nS'$ that is if $m, n \in S$ generate same right ideal. It is $m\ \mathcal{J}\ n$ if and only if $S'm\ S'$ or $S'n\ S'$ that is if $S'm\ S' = S'n\ S'$. (generate two-sided ideal). It is $\mathcal{H}\ m$ that is the intersection of $\mathcal{L}\ m$ and $\mathcal{R}\ m$. More generally the intersection of any $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$ is either an $\mathcal{H}_{class}$ or the empty set. That is $\mathcal{H}\ m$ is either $\mathcal{H}^2 \cap \mathcal{H} = \emptyset$ or $\mathcal{H}^2 = \mathcal{H}$ such that $\mathcal{H} \subseteq S \subseteq CP_n$.It is $\mathcal{D}\ m$ that is if $m\ \mathcal{L}\ n$ and $m\ \mathcal{R}\ n$ such that $\mathcal{L}\ \upsilon\mathcal{R}$ is the minimum equivalence relation class containing $\mathcal{L}\ \upsilon\mathcal{R}$ and other equivalence class. From the above we have the formulation such that $m\ \mathcal{D}\ n$ if and only if the intersection $\mathcal{R}m\ \cap \mathcal{L}n \neq \emptyset$. But if otherwise, then either every element of $\mathcal{D}_{class}$ is regular or no element of $\mathcal{D}_{class}$ is regular. We called the $\mathcal{D}_{class}$ regular if all its elements are regular and irregular if otherwise. Consequently, the $\mathcal{D}_{class}$ can be seen as the union of $\mathcal{L}_{class}$, $\mathcal{R}_{class}$ and $\mathcal{H}_{class}$. We depict the content of our results by means of what [8] refer to as Egg-box picture. In such a picture, each $\mathcal{D}_{class}$ of S was represented by square which is divided into vertical rectangles (columns) representing $\mathcal{L}_{class}$ and the horizontal rectangles (rows) representing $\mathcal{R}_{class}$. Each small square formed by the intersection of such rectangle represents the $\mathcal{H}_{class}$. Viz.

The cells in the above table give all $\mathcal{H}_{class}$ contained in our $\mathcal{D}_{class}$. We observed that all cells are non-empty (Tables 4.1 and 4.2).

Since empty set is a sub-set of every set (M_i *say*); for it is not, it means there is an element of empty set that is not in M_i. This is contrary to the definition of empty set. In the current manuscript, we conclude the empty set $\emptyset$ in the transformation. Let $(--)$ stands for $\emptyset$ and let $\begin{pmatrix} a & b & c \\ m & n & p \end{pmatrix}$ be represented by $(m\ n\ p)$ not (m,n,p) and not complicating the cycle notation. The mapping that included the empty set $\emptyset$ is referred to as partial transformation P_n. The trivial fact that the composition of functions is associative gives rise to one of the most promising families of semi-groups (transformation) for the present and next generation of researchers [9].

Table 4.1
Order of P_n elements in the five class relation.

P_5	$\mathcal{L}_n$	$\mathcal{R}_n$	$\mathcal{H}_n$	$\mathcal{D}_n$	$\mathcal{J}_n$	$\lvert\alpha(S)\rvert = \sum_{n=0}^{5} P_n$
0	1	1	1	1	1	5
1	2	2	2	2	2	10
2	6	6	3	3	3	21
3	26	14	6	4	4	54
4	142	30	16	5	5	198
5	912	62	53	6	6	1093

Table 4.2
Order of CP_n elements in the five class relation.

CP_5	$\mathcal{L}_n$	$\mathcal{R}_n$	$\mathcal{H}_n$	$\mathcal{D}_n$	$\mathcal{J}_n$	$\lvert\alpha(S)\rvert = \sum_{n=0}^{5} CP_n$
0	1	1	1	1	1	5
1	2	2	2	2	2	10
2	6	3	6	3	3	21
3	13	6	23	4	4	50
4	20	14	92	5	5	136
5	37	50	?	6	6	?

Let T_n be the set of full transformations (mappings) and P_n be the set of partial transformations. Then T_n forms a semi-group of order n^n and P_n forms a semi-group of order $(n+1)^n$ [7]. Let S_n be the symmetric sub-group of the transformation semi-group T_n. Then $|S_n| = n!$. The n! is always less than n^n, when $n > 1$. Refer to [10, 11] for order symmetric semi-group $|S_n|$ and for the order of the alternating sub-group of the symmetric group $|A_n|$.

Let $M = \{1, 2 \ldots\}$ be the set of n number of ordered elements of counting numbers representing symbols of mathematical intuitions and thought. Let α be a mapping from sub-sets of M to sub-sets M. Then the set of all αS if equipped with the binary operation of composition of mappings forms the partial transformation semi-group [12]. In [7] contraction mapping in P_n is defined as: for all $m, n \in D(\alpha)$, $|\alpha m - \alpha n| \leq |m - n|$. The breadth of α is denoted by $b(\alpha) = |D(\alpha)|$. The height of α is denoted by $h(\alpha) = |I(\alpha)|$. The fix of α is given by $f(\alpha) = \{\alpha(m) = m \ for \ all \ m \in S\}$ and the collapse is denoted by $c(\alpha)$.

A transformation $\alpha \in P_n$ is said to be order-preserving if for all $m, n \in D(\alpha) : m \leq n$ implies $\alpha m \leq \alpha n$. It is order-reversing if $m \leq n$ implies $\alpha m \geq \alpha n$. It is order-decreasing if $\alpha m \leq m$. It is order-preserving or reversing if $m \leq n$ implies $\alpha m \leq \alpha n$ union $m \leq n$ implies $\alpha m \geq \alpha n$ and it is order-preserving and decreasing if $m \leq n$ implies $\alpha m \leq \alpha n$ intersection $m \leq n$ implies $\alpha m \leq m$.

The following notion will assist to understand the concept of contraction mapping in Mathematical term. Let $|CP_n|$ be the order of set of all contraction mappings of P_n, for a (partial) $\alpha \in P_n : \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ such that $D(\alpha) = (1\ 2\ 3\ 4)$ and $I(\alpha) = (1\ 1\ 1\ 2)$ then we have $|\alpha m - \alpha n|$ *and* $|m - n|$ that is $|\alpha m - \alpha n| \leq |m - n|$ implies $D(\alpha) \subseteq M$ whenever $m, n \in D(\alpha)$. It is trivial to show for all $m, n \in D(\alpha)$, α satisfy contraction inequalities such that α is a contraction mapping.

The main purpose of the current manuscript is to characterize the five equivalence relations class on the above mentioned semi-groups $S \subseteq P_n$.The manuscript is organized: In Section 'Basic preliminaries', we define some basic preliminaries and present some results from past literatures. In Section 'The five equivalence relation classes on CP_n', we depict the $\mathcal{D}_{class}$ of CP_n, obtain some theorems and consider $|\alpha_{ij}(S)|$ on the equivalence class. In Section 'Concluding remarks', we give summary chart of the number of elements of each equivalence class with some concluding remarks.

Basic preliminaries

In this section, we present some basic definitions and results concerning the five equivalence relation class and contraction transformation semi-groups:

Definition 2.1. [Domain of Relation $\mathcal{D}(\boldsymbol{\rho})$]: Let ρ be a relation from a set M into a set N. Then the domain of $\boldsymbol{\rho}$, $\mathcal{D}(\boldsymbol{\rho})$ is defined to be the set $\{m | m \in M \ and \ there \ exists \ n \in N : (m, n) \in \boldsymbol{\rho}\}$.

Definition 2.2. [Image of Relation $\mathcal{I}(\boldsymbol{\rho})$]: Let ρ be a relation from a set M into a set N. Then the image set of $\boldsymbol{\rho}$, $\mathcal{I}(\boldsymbol{\rho})$ is defined to be the set $\{n | n \in N \ and \ there \ exists \ m \in M : (m, n) \in \boldsymbol{\rho}\}$.

Definition 2.3. [Equivalence Relation $(\equiv \boldsymbol{\rho})$]: A binary relation $\equiv \boldsymbol{\rho}$ on a set M is called an equivalence relation if $\equiv \boldsymbol{\rho}$ is reflexive, symmetric and transitive.

Definition 2.4. [Equivalence Class $[\boldsymbol{m}]$]: Let $\equiv \boldsymbol{\rho}$ be an equivalence relation on a set M. For all $m \in M$, let $[\boldsymbol{m}]$ denote the set $[\boldsymbol{m}] = \{n \in M | n \equiv \boldsymbol{\rho} m\}$**.** Then the set $[\boldsymbol{m}]$ is called the equivalence class with respect to $\equiv \boldsymbol{\rho}$ determined by m.

Definition 2.5. [Mapping $(\rightarrow)$]: Let M and N be non-empty sets. A relation $\boldsymbol{\rho}$ from M into N is called a mapping from M into N if:

i $\mathcal{D}(\boldsymbol{\rho}) = \boldsymbol{M}$
ii For all $(m, n), (k'l') \in \boldsymbol{\rho}$**,** $m = k$ implies $n = l$.

When (ii) is satisfy by $\boldsymbol{\rho}$**,** we say $\boldsymbol{\rho}$ is well defined. We use the notation $\boldsymbol{\rho} : M \rightarrow N$ denote a mapping from set M into N. For $(m, n) \in \boldsymbol{\rho}$, we usually write $\boldsymbol{\rho}(\boldsymbol{m}) = \boldsymbol{n}$ and say n is the image of m under $\boldsymbol{\rho}$ and m is pre-image of n. Suppose $\boldsymbol{\rho} : \boldsymbol{M} \rightarrow \boldsymbol{N}$. Then $\boldsymbol{\rho}$ is a sub-set of M X N such that for all $m \in M$, there exists a unique $n \in N$ such that $(m, n \in \boldsymbol{\rho})$. Hence, we like to think mapping as a rule which associates to each element $m \in M$ exactly one element $n \in N$. If a relation $\boldsymbol{\rho}$ is a transformation then the domain $\mathcal{D}(\boldsymbol{\rho})$ of $\boldsymbol{\rho}$ is M such that $\boldsymbol{\rho}$ is well defined that if $m = n \in M$, then $\boldsymbol{\rho}(\boldsymbol{m}) = \boldsymbol{\rho}(\boldsymbol{n}) \in N$ for all $m, n \in M$.

Definition 2.6. [Compatible $(\boldsymbol{m\rho n})$]: Let S be a semi-group, a relation $\boldsymbol{\rho}$ on S is said to be left (right) compatible if $m\boldsymbol{\rho}n$ implies $Sm\boldsymbol{\rho}Sn$ and $(mS\boldsymbol{\rho}nS)$ for all $m, n \in S$.

Definition 2.7. [Partial Mapping]: An (transformation) α whose not all the element in the domain $\mathcal{D}(\alpha)$ has image set $\mathcal{I}(\alpha)$ is refers to as partial mapping of a finite semi-group S.

Let $M = \{\emptyset\}$ empty map, a partial mapping $\alpha : M \rightarrow M$ is a rule that take from a sub-set $N = \mathcal{D}(\alpha)$ of M onto $\mathcal{I}(\alpha) = \alpha(N) \subseteq M$. Let $M = \{1, \ldots 5\}$ and let $\mathcal{D}(\alpha) = \{2, 4, 5\}$ with $\alpha(2) = 1$, $\alpha(4) = 5$ and $\alpha(5) = 1$. We can represent α viz: $\alpha = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 5 & 1 \end{pmatrix}$, therefore $\mathcal{I}(\alpha) = \{1, 5\}$.

Definition 2.8. [Regular Element $(\boldsymbol{R_e})$]: An element $m \in S$ where S is a finite semi-group is said to be regular if there exists a unique element $n \in S$ such that $mnm = m$.

Definition 2.9. [Semi-group $(\boldsymbol{S})$]: A non-empty set S with a binary operation $(m, n) \rightarrow mn$ from $S * S$ to S is said to be a semi-group if there exists k such that $m(nk) = (mn)k$ for all $m, n, k \in S$.

Definition 2.10. [Sub-Semi-group $(\boldsymbol{S_b})$]: A non-empty set M of a semi-group S is said to be a sub-semi-group of S if M itself is a semi-group as in [2.9] induced on S. That is M is a sub-semi-group if and only if $mn \in M$ for all $m, n \in M$.

Definition 2.11. [Monoid $(\boldsymbol{S})$]: A semi-group S is said to be a monoid if there exists an element $1 \in S$ with $m1 = 1m = m$ for $m \in S$. The element 1 which is necessarily unique is called identity of S.

Definition 2.12. [Regular Semi-group $(\boldsymbol{S_r})$]: A semi-group S in which every element is regular is called a regular semi-group such that $I_m = \{n : mnm = m\}$ for all $m, n \in S$.

Definition 2.13. [Simple Semi-group $(\boldsymbol{S_p})$]: A simple semi-group **$(\boldsymbol{S_p})$** is a semi-group that has only one $\mathcal{J}_{class}$. That is it has non-trivial element.

Definition 2.14. [Ideals (*)]: A non-empty subset $M$of a semi-group S is left ideal if $SM \subseteq M$, it is right ideal if $MS \subseteq M$, and it is two-sided ideal if it is both a left and left ideal.

Definition 2.15. [Equivalence on a Semi-group (S)]: Let S be a semi-group we define the five equivalence relation classes on S:

$$\left.\begin{array}{c} \mathcal{L} = \{(m, n) \in S * S : S'm = S'n\} \\ \mathcal{R} = \{(m, n) \in S * S : mS' = nS'\} \\ \mathcal{J} = \{(m, n) \in S * S : S'mS' = S'nS'\} \\ \mathcal{H} = \mathcal{L} \cap \mathcal{R} \\ \mathcal{D} = \mathcal{L} \upsilon \mathcal{R} \end{array}\right\} \tag{2.15}$$

From the above definitions, we have the inclusion diagram which [13] refers to as Hasse diagram (Fig. 1.1).

A Hasse diagram of the five equivalence relation classes $\subseteq$ on a mapping CP_n is a diagrammatic representation of $\subseteq$. Every classes represented by a point on the plane, arranged so that $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$ appears below $\mathcal{D}_{class}$ whenever $\mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{R} \subseteq \mathcal{D}$. We observe that $\mathcal{H}_{class}$ is an equivalence relation and it is the largest equivalence of the five class of S that is contained in both $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$ and $\mathcal{D} \subseteq \mathcal{J}$ by Fig. 2.16. The result follows immediately.

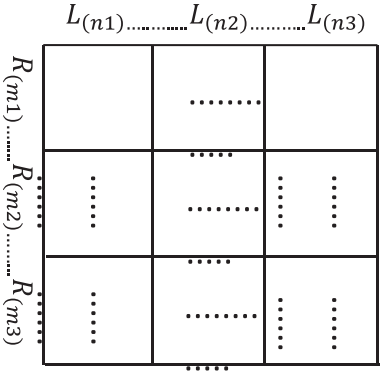


Fig. 1.1.

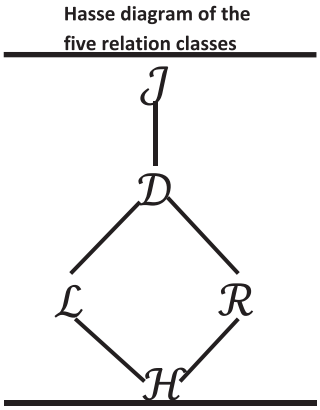


Fig. 2.16.

Proposition 2.17. *The relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$ and $\mathcal{D}$ are equivalence on S.*

Proof: Suppose $m, n \in S$ by definition,
$m\mathcal{L}n$ if and only if there exist $k, l \in S'$; $m = kn$ and $n = lm$
It is $m\mathcal{R}n$ if and only if there exist,$n \in S'$; $m = nk$ and $n = ml$
It is $m\mathcal{J}n$ if and only if there exist,$n, u, v \in S'$; $m = knl$ and $n = umv$
Similarly, by definition [2.15], we observed that $\mathcal{L} \upsilon \mathcal{R} = D$. Hence, the result follows immediately from Fig. 2.16. 2.17

Definition 2.18. [Idempotent Element (m)]: An element $m \in S$ of a finite semi-group is idempotent if $m^2 = m$. In a transformation semi-group, an element $\alpha \in S$ is idempotent if and only if $I(\alpha) = F(\alpha)$, where $F(\alpha)$ is the set of order of the elements that maps to itself only (fixed point). That is $F(\alpha) = |\{m \in D(\alpha) : m\alpha = m\}|$. It is denoted by $F(\alpha) = m$, and $I(\alpha)$is the set of image of the transformation.

Definition 2.19. [Partial Transformation P_n]: Let $M = \{1, 2 \ldots\}$ be a natural ordering of numbers and $\alpha : D(\alpha) \subseteq M$ implies $I(\alpha) \subseteq M$. Then partial transformation P_n is the set of all functions $\alpha : D(\alpha) \subseteq M$ on M.

Definition 2.20. [Partial Contraction Mapping CP_n]: Let $M = \{1, 2 \ldots\}$. Then a transformation $\alpha \in P_n$ is said to be partial contraction mapping if for all $m, n, k \in CP_n$ *implies* $D(\alpha)$: $|\alpha m - \alpha n| \leq |m - n|$ satisfying $m \mathcal{L} n \mathcal{R} k$, then there exists an idempotent $e \in S$ such that $m \mathcal{R} e \mathcal{L} k$.

Definition 2.21. [Breadth / Width, (α)]: This is the number of elements in the domain of α. That is $|D(\alpha)| = b(\alpha)$.

Definition 2.22. [Height / Length, $h(\alpha)$]: This is the number of elements in the image sets of α. That is $h(\alpha) = |I(\alpha)|$ and it is denoted by $h(\alpha) = p$.

Definition 2.23. [Collapse, $c(\alpha)$]: This is the order of the union of image sets of α which is greater than 2. That is $c(\alpha) = |\cup\{q(\alpha)^{-1} : q \in I(\alpha) and\ |q(\alpha)^{-1}| \geq 2\}|$.

Definition 2.24. [Right waist, $\omega^+(\alpha)$]: This is the maximum element in the image sets of $(I(\alpha))$ of α. That is $\omega^+(\alpha) = \max(I(\alpha))$. It is denoted by $\omega^+(\alpha) = k$.

Definition 2.25. [Left waist, $\omega^-(\alpha)$]: This is the minimum element in the image sets of $(I(\alpha))$ of α. That is $\omega^-(\alpha) = \min(I(\alpha))$. It is denoted by $\omega^-(\alpha) = k^-$.

Definition 2.26. [Identity Element (S_e)]: If there exist an element $1 \in M$ of a semi-group of S (monoid) such that $1m = m1 = m$ for all $m \in S$. It means that 1 is an identity element of S. Thus there exist $e \in S$ such that $me = em = m$ for all $e, m \in S$.

Definition 2.27. [Nilpotent Element (S_n)]: An element $m \in S$ (monoid) with zero is a nilpotent if there exist a positive $k \in Z^+$ such that $m^k = 0$ for all $m, k \in S$. Similarly, a partial transformation α is nilpotent if $\alpha^k = \emptyset$.

Definition 2.28. [Inverse Semi-group (S_I)]: A semi-group S is called an inverse semi-group if every $m \in S$ posses a unique inverse. That is if there exists a unique $m^{-1} \in S$ such that $mm^{-1}m = m$ and $m^{-1}mm^{-1} = m^{-1}$.

Definition 2.29. [Universes (U_e)]: In a finite Semi-group S, two elements m and n of S are said to be universes $[U_e]$ for all $m, n \in U_e$ such that $U_e \subseteq S$ of each other if $mnm = m$ and $nmn = n$.

Lemma 2.30. *Every regular element $m_{(i)} \in S$ has at least one inverse.*

Proof: Since $m = mnm$, for all $m, n \in S$

If $p = nmn$ then $mpn = m(nmn)m = (mnm)nm$

$$= m(n)m = m$$

$$= m*(pmp)$$

$$= n(mnm) = (nmn)(mnm)n = p$$

Hence, m and p are universes, for all $n, m, p \in S$. 2.30

The five equivalence relation classes on CP_n

In this section, the five equivalence relation is characterize on the more general semi-group of partial contraction mapping CP_n. The class defined in Section 'Basic preliminaries' are collectively called **Greens relations.** For $m \in S$, the $\mathcal{L}_{class}$, $\mathcal{R}_{class}$, $\mathcal{H}_{class}$ and $\mathcal{J}_{class}$ containing m will be denoted by $\mathcal{L}_m$, $\mathcal{R}_m$, $\mathcal{H}_m$, and $\mathcal{I}_m$ respectively. Since $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are define in terms of principal ideals, we can define partial orders on $S/\mathcal{L}$, $S/\mathcal{R}$, $S/\mathcal{J}$ by:

$\mathcal{L}_m \leq \mathcal{L}_n$ if and only if $S'_m \subseteq S'_n$
$\mathcal{R}_m \leq \mathcal{R}_n$ if and only if $mS' \subseteq nS'$
$\mathcal{J}_m \leq \mathcal{J}_n$ if and only if $S'_m S' \subseteq S'_n S'$.

We show that these class are equivalence relations, and that $\mathcal{D}_{class}$ is regular and coincides with $\mathcal{J}_{class}$. Hence, a semi-group S is said to be contraction if for all $m, n, k \in CP_n$ satisfying $m \mathcal{L} n \mathcal{R} k$ then, there exists an idempotent $e \in S$ such that $m \mathcal{R} e \mathcal{L} k$. Therefore, by [2.17, 2.18] $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J}$.

In the current manuscript, we present the egg-box diagram for $\mathcal{D}_{class}$ of the semi-group CP_n. For example, the transformation $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ from CP_3 is given by the second row $(m_1\ m_2\ m_3) \neq (m_1, m_2, m_3)$. CP_n has four $\mathcal{D}_{class}$ $\mathcal{D}_i = \{\emptyset,\ 1,\ 2,\ 3\}$ by the rank of elements inside the class. To accommodate the picture to the page size we transpose α_i. So in our picture rows are $\mathcal{R}_{class}$ and columns are $\mathcal{L}_{class}$. Let $M = \{1, 2 \ldots\}$, then a transformation $\alpha \in P_n$ on the set M is partial contraction

mapping if for all $m, n \in D(\alpha)$ such that $|m\alpha - n\alpha| \leq |m - n|$. The contraction elements of CP_n for $n = 3$ was listed in the $\mathcal{D}_{class}$ from which we observe the $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$.Viz:

$\mathcal{D}_0$: $\emptyset$

$(\emptyset \ \emptyset \ \emptyset)$

$\mathcal{D}_1$: {1} {2} {3}

(1 1 1)	(2 2 2)	(3 3 3)
(1 1 ∅)	(2 2 ∅)	(3 3 ∅)
(1 ∅ 1)	(2 ∅ 2)	(3 ∅ 3)
(∅ 1 1)	(∅ 2 2)	(∅ 3 3)
(1 ∅ ∅)	(∅ 2 2)	(∅ 3 3)
(∅ 1 ∅)	(∅ 2 ∅)	(∅ 3 ∅)
(∅ ∅ 1)	(∅ ∅ 2)	(∅ ∅ 3)

$\mathcal{D}_2$: {1, 2} {1, 3} {2, 3}

(1 ∅ 2)(2 ∅ 1)	(1 ∅ 3)(3 ∅ 1)	(2 ∅ 3)(3 ∅ 2)
(∅ 1 2)(∅ 2 1)		(∅ 2 3)(∅ 3 2)
(1 2 ∅)(2 1 ∅)		(2 3 ∅)(3 2 ∅)
(1 1 2)(2 2 1)		(2 2 3)(3 3 2)
(1 2 2)(2 1 1)		(2 3 3)(3 2 2)
(1 2 1)(2 1 2)		(2 3 2)(3 2 3)

$\mathcal{D}_3$: {1,2,3}

(1 2 3) (3 2 1)

From the tables, the $\mathcal{L}_{class}$, $\mathcal{R}_{class}$ and $\mathcal{H}_{class}$ for all $m, n \in D(\alpha)$ we have α such that $D(\alpha) \subseteq M$ implies $I(\alpha) \subseteq M$. Given that $\beta, \alpha \in S$ where $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, we show that α and β are five equivalence related viz:

	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
$\mathcal{R}_0$	(∅ ∅ ∅)												
$\mathcal{R}_1$		(1 1 1)	(2 2 2)	(3 3 3)									
$\mathcal{R}_2$					(1 1 ∅)	(2 2 ∅)	(3 3 ∅)				(1 1 2)	(2 2 3)	
					(∅ ∅ 1)	(∅ ∅ 2)	(∅ ∅ 3)				(2 2 1)	(3 3 2)	
$\mathcal{R}_3$					(1 ∅ ∅)	(2 ∅ ∅)	(3 ∅ ∅)				(1 2 2)	(2 3 3)	
					(∅ 1 1)	(∅ 2 2)	(∅ 3 3)				(2 1 1)	(3 2 2)	
$\mathcal{R}_4$					(1 ∅ 1)	(2 ∅ 2)	(3 ∅ 3)				(1 2 1)	(2 3 2)	
					(∅ 1 ∅)	(∅ 2 ∅)	(∅ 3 ∅)				(2 1 2)	(3 2 3)	
$\mathcal{R}_5$								(1 2 ∅)		(2 3 ∅)			
								(2 1 ∅)		(3 2 ∅)			
								(1 ∅ 2)	(1 ∅ 3)	(2 ∅ 3)			
								(2 ∅ 1)	(3 ∅ 1)	(3 ∅ 2)			
								(∅ 1 2)		(∅ 2 3)			(1 2 3)
								(∅ 2 1)		(∅ 3 2)			(3 2 1)

Two transformations are $\mathcal{L}_{related}$ if and only if they have same image, its $\mathcal{R}_{related}$ if and only if for $\alpha, \beta \in M$ such that are in same table row and its $\mathcal{D}_{related}$ if and only their image have the same size. Since $\mathcal{H}_{class}$ is the largest equivalence relation that contained both the $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$ and $\mathcal{D} \subseteq \mathcal{J}$. Hence, the corollary follows immediately.

Corollary 3.1. *If $\alpha, \beta \in S$, for $\alpha \ \mathcal{L} \ \beta$ and $\beta \ \mathcal{R} \ \alpha$, then no $\mathcal{H}_{class}$ in regular $\mathcal{D}_{class}$ contains more than one inverse of $m \in \mathcal{D}$.*

Proof: Suppose m° and m^* universes, and for all $a, m^\circ, m' \in \mathcal{H}$, implies mm^* and mm° are both idempotent in $\mathcal{H}_{class}$ such that $\mathcal{R}_m \cap \mathcal{L}_n$ implies $mm^* = mm^\circ$. But $\mathcal{H}_e$ containing m such that idempotent $e \in S$ is a sub-group and by the five equivalence relations theorem no $\mathcal{H}_{class}$ can contain more than one idempotent implies $mm^* = mm^\circ$ implies $m^*m = m^\circ m$; $m, n \in \mathcal{L}_m \cap \mathcal{R}_n$. Hence, $m^\circ = m^\circ mm^\circ = m^\circ mm^* = m^*mm^* = m^*$. The proof is complete. 3.1

Theorem 3.2. *let k be an integral domain, and $(a, s) \in K \ X \ S$ such that $S \neq 0$ and $(a, s)\rho(b, t)\lambda$ if and only if $(at - bs)\lambda = 0$ then ρ is equivalence.*

Proof: A relation ρ is an equivalence relation on set (a, s) if:

i $(a, s)\rho(a, s)$
ii $(a, s)\rho(b, t) = (b, t)\rho(a, s)$
iii $(a, s)\rho(b, t)\rho(c, u)$ implies $(a, s)\rho(c, u)$

Since $a\ s - a\ s = 0$ implies $(a, s)\rho(a, s)$. Therefore ρ is reflexive.

If $(a, s)\rho(b, t)$ implies $a\ t - b\ s = 0$ then $--(at - bs) = 0$ implies $(a, t)\rho(a, s)$ implies $b\ s - a\ t$ which gives $(a, s)\rho(b, t) = (b, t)\rho(a, s)$. Therefore ρ is symmetric. Similarly, if $(a, s)\rho(b, t)\rho(c, u)$ implies $(a, s)\rho(c, u)$. Since $(a, s)\rho(b, t) = a\ t - b\ s = 0$ $(*)$ and $(b, t)\rho(c, u) = b\ u - c\ t = 0$ $(**)$

Multiply $(*)$ by u and $(**)$ by s we have $a\ t\ u - b\ s\ u$ and $b\ u\ s - c\ t\ u$ implies $a\ t\ u = c\ t\ s$ implies $a\ u - c\ u$ since $t \neq 0$ implies $(a, s)\rho(c, u)$. Therefore ρ is transitive. Hence, the proof is complete. 3.2

Theorem 3.3. *If $a \in S$ is a regular element such that $a\ x\ a = a$, then ax and xa are idempotent.*

Proof: Since a is regular, we need to show that ax is idempotent implies $(ax)^2 = ax$. Then $(ax)^2 = (ax)(ax) = (a\ x\ a)x = ax$.

Similarly, $(xa)^2 = xa$. Hence the result. 3.3

Proposition 3.4. *If $a \in S$ is regular if and only if $eS' = aS'$; then a is idempotent and has a unique generator e.*

Proof: Suppose $a \in S$ is regular, $a\ x = x\ a$

Implies $a\ x\ a = a$ for all $a, x \in S$ then, there exists $e = a\ x$ implies $e\ a = a\ x\ a = a$ implies $aS' = eS'$ for all $e^2 = e$.

Hence, $eS' = aS'$.

Conversely, we have $eS' = aS'$ if $e^2 = e$, implies $a = e\ x$ for $x\ \in S'$ implies $e\ a = \ e^2\ x$ implies $e\ x = a$. Then, $a^2 = a,\ a^3 = a, \ldots a^n = a$. Therefore, a is regular. 3.4

Theorem 3.5. *If $n \in X_m : \{n \in X_m | n\ x\ n = n; nx = xn\}$ then $n \in \mathcal{D}_{class}$ is regular.*

Proof: Suppose $n\ \mathcal{D}\ m$ such that $n, m \in \mathcal{D}$, implies there exists k such that $n\ \mathcal{L}\ k$ and $k\ \mathcal{R}\ m$ for $n, m, k \in S$. Let there exists $u, v, s, t \in S$, since $n\ \mathcal{L}\ k = S'n = S'k$, implies $k = n\ u$ implies $n = n\ v\ k$, also, $k\ \mathcal{R}\ m = \ kS' = mS'$ implies $m = k\ s$ which gives $k = m\ t$.

Suppose also $n \in X_m$, implies $n\ x\ n = n$ for $x \in S$. Now, let $m' = t\ x\ v$ implies $m\ m'm = m(t\ x\ v)b = k\ x\ v\ k\ s = k\ x\ n\ s$

$$= u(n\ x\ n\ )s = u\ n\ s\ k\ s = m.$$

Therefore, by proposition [3.4] $m \in X_m : m\ x\ m = m$. 3.5

Theorem 3.6. *The relation $\mathcal{L}_{class}$ on S satisfies $n\ \mathcal{L}\ k$ if and only if there exists $p, q \in S'$such that $p\ n = k$ and $q\ k = n$.*

Proof: Suppose $n\ \mathcal{L}\ k$, then by [2.17], $S'n = S'k$. Since $k \in S'k$, it follows that $k \in S'n$ and so there exists $p \in S'$ such that $p\ n = k$. Similarly, there exists $q \in S'$such that $q\ k = n$. Now, suppose that there exists $p, q \in S'$ such that $p\ n = k$ and $q\ k = n$. Then $S'n = S'q\ k\ \subseteq S'k$ and similarly, $S'k = S'p\ n \subseteq S'n$. Hence, $S'n = S'k$ and so $n\ \mathcal{L}\ k$. Then by (2.15) and Fig. 2.16 the results follows similarly for the $\mathcal{R}_{class}$ and $\mathcal{J}_{class}$. 3.6

Theorem 3.7. *In any finite semi-group S, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, consequently, for $m, n \in S$ the following axioms are equivalent.*

i $m\ \mathcal{D}\ n$
ii There exists $k \in S$ with $m\ \mathcal{L}\ k\ \mathcal{R}\ n$
iii There exists $k \in S$ with $m\ \mathcal{R}\ k\ \mathcal{L}\ n$

Proof: (i) $\Rightarrow$ (ii) Let $= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Suppose the relation $\mathcal{L} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{L}$ is satisfy such that there exists $m,\ n \in S$ whenever $m\ \mathcal{L}\ n$ if and only if there exist $k, l \in S'$ where $((km = n) \wedge (ln = m))$. Then If $\mathcal{L} \circ \mathcal{R}$ implies $m\ \mathcal{L}\ n\ \circ k\ \mathcal{R}\ l$ for $m, n, k, l \in S'$ such that $(m, n) \in \mathcal{L}$ and $(k, l) \in \mathcal{R}$. Implies $(mk,\ nl) \in \mathcal{L} \circ \mathcal{R} = \mathcal{H}$, then by proposition [2.17] we have that if $(m, n) \in \mathcal{L} \circ \mathcal{R}$ if there exists $k \in S$ such that $m\ \mathcal{L}\ k$ and $k\ \mathcal{R}\ n$.

(ii) $\Rightarrow$ (iii) Let there exists also $p, q, u, v \in S'$ such that

$$\left.\begin{array}{ll} p\ m = k; & k\ u = n \\ k\ u = n; & n\ v = k \end{array}\right\} \tag{3.7.1}$$

From [3.7.1], we have that $k\ u = n$ implies $k = n\ u^{-1}$, similarly, $q\ k\ n = m\ k\ u$ implies $m\ u = q\ k\ u = l$. Now, let $l = q\ k\ u$ we have $q\ k\ u = q\ n\ u = p\ m\ k = n\ v\ k = p\ m = k$. We need to show that $t\ v = q\ k\ u\ v = q\ n\ v = q\ k = m$. Then if $q\ k = m$; $n\ v = k$ implies $q\ k\ k = m\ n\ v$ implies $q\ n\ v\ k = m\ n\ v$ implies $q\ k = m$.

$$\left.\begin{array}{l} \text{Therefore, } m\ u = q\ k\ u = l \text{ and} \\ \qquad\qquad q\ k = m \end{array}\right\} m\mathcal{R}\ l \tag{3.7.2}$$

Also, we need to show $q\ n = q\ k\ u = l$. Similarly, from [3.7.1] $q\ k = m$; $n\ v = k$, implies $q\ k\ n = m\ k\ u$ implies $q\ k\ n = q\ k\ u$, implies $q\ k\ k\ u = q\ k\ k\ u$.

Then, $q\ n = q\ k\ u = l$. Also, $k\ u = p\ m\ u = n = p\ l = p\ q\ k\ u$ implies $m\ p = k = k\ u = n$ which gives $k\ k\ u = m\ p\ n$ implies $k\ k\ u = m\ p\ k\ u$ implies $k\ u = m\ p\ u = n$.

$$\left.\begin{array}{l} \text{Therefore, } p\ n = p\ k\ u = l \\ \qquad\qquad k\ u = m\ p\ u = n \end{array}\right\} l\mathcal{L}\ n \tag{3.7.3}$$

(iii) $\Rightarrow$ (i) Since $m\ \mathcal{D}\ n$ such that for all $m,\ n\ \in S$, finite semi-group then $m, n \in \mathcal{L} \circ\ \mathcal{R}$ implies $\mathcal{L} \circ\ \mathcal{R} \subseteq \mathcal{R} \circ\ \mathcal{L}$. Then, from the above theorem, we observe $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L}\ \upsilon\mathcal{R}$ implies $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Therefore, $\mathcal{D}$ is the minimum equivalence relation class containing $\mathcal{L}$ and $\mathcal{R}$. Thus, $\mathcal{D} \subseteq \mathcal{J}$ follows immediately by theorem [3.5]. Hence the proof is complete. 3.7

Lemma 3.8. *$\mathcal{D}_{class}$ is the minimum equivalence class that contains both $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$.*

Proof: Suppose $m, n \in S$, implies $m\ \mathcal{L}\ n$ and $m\ \mathcal{R}\ n$. By theorem [3.7] $\mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{R} \subseteq \mathcal{D}$ and since $m\ \mathcal{L}\ n$ and $m\ \mathcal{R}\ n$ holds for $m, n \in S'$. In order to show $\mathcal{D}_{class}$ is the minimum equivalence class that contains both $\mathcal{L}_{class}$ and $\mathcal{R}_{class}$, we needs show that it is transitive such that if there exists $p \in S$ we have $m\ \mathcal{D}\ p$ and $p\ \mathcal{D}\ n$ then $m\ \mathcal{D}\ n$.

Conversely, suppose $\mathcal{D}$ is the minimum equivalence containing $\mathcal{L} \upsilon \mathcal{R}$. since $\mathcal{L}$ and $\mathcal{R}$ are equivalence by theorem [3.6] implies $m \mathcal{L} n$ and $m \mathcal{R} n$ for all $m, n, p \in S$. Then exists an equivalence relation ρ with $\mathcal{L} \upsilon \mathcal{R} \subseteq \rho$ implies $\mathcal{D} \subseteq \rho$. Since ρ_{max} containing $\mathcal{D}$ is an equivalence relation and $\mathcal{D}$ has non-trivial element of S and empty $\{\emptyset\}$ is ρ then any non-trivial element ζ must contain $\{\emptyset\}$. Hence, such element doest not exists since ρ is the maximal and contain $\mathcal{L} \upsilon \mathcal{R} \subseteq \rho$ implies $\mathcal{D} \subseteq \rho$. The proof is complete by proposition (2.15) and Fig. 2.16. 3.8

Proposition 3.9. *Let m be an idempotent and there exists n, p such that $m, n, p \in S$, then $\mathcal{L}_m \leq \mathcal{L}_n$ if and only if $m\ n = m$ and $\mathcal{R}_m \leq \mathcal{R}_n$ if $n\ m = m$.*

Proof: Since $m, n \in S$ implies $m, n \in \mathcal{L}_{(m,n)}$ and $\mathcal{R}_{(m,n)}$.

Then, in particular for $n, p \in E(S)$, then $n \mathcal{L} p$ if and only if $n\ p = n$ and $p\ n = p$. Also, $n \mathcal{R} p$ if and only if only if $n\ p = p$ and $p\ n = n$ and $n \mathcal{H} p$ if and only if $n = p$. 3.9

Theorem 3.10. *If $m \in S$ such that $M = \{m | m \in M; m\ n\ m = m\}$if and only if $m\ S'n = n\ S'm$ and there exists $e \in S$.*

Proof: Suppose m is regular and the principal left (right) ideal generated by m has an idempotent generator e then $m = m\ n\ m$ for some $n \in S$. Hence, $e = m\ n$ is an idempotent with $e\ m = m\ n\ m = m$ implies $eS' = mS'$.

Conversely, $mS' = eS'$ such that $e^2 = e$, then $m = e\ n$ for some $m \in S'$. Hence, $e\ m = e^2 n = e\ n = m$. Also, $e = m\ k$ for some $k \in S'$. Thus, $m = e\ m = e\ k\ m$ such that $k = 1$, then by theorem [3.4] $m = m^2$ and $m^3 = m * m = m$. Therefore, in any case m is regular. 3.10

Concluding remarks

In this section, we present an overview of the number of elements in each equivalence relation class on the Semi-group of P_n and its sub-semi-group CP_nViz:

Remark 4.1. For all the semi-group presented, $f(n, m) = 1$ whenever $n = m$. Similarly, $f(n, k) = 1$ whenever $k = 0$ and $|P_n| = |CP_n| = 2$). However, there exist bijection between the class of the relations on both semi-groups for $n \geq 2$.

Remark 4.2. Two transformation α and β are in $\mathcal{L}_{class}$ if and only if they have the same set of image that is $(\alpha\beta) \in \mathcal{L}$ if and only if rank (α) = rank (β). It is in $\mathcal{R}_{class}$ if and only if $\alpha * \alpha^{-1} = \beta * \beta^{-1}$ such that $\alpha * \alpha^{-1} = \{(m, n) \in \alpha : \alpha(m) = \alpha(n)\}$ and It is in $\mathcal{D}_{class}$ if and only if rank $|r(\alpha)|$ = rank $|r(\beta)|$

Remark 4.3. The transformation semi-group was the most promising class of semi-groups for future study [9]. The forecast was justified because many researchers have worked extensively on the subject. Such as [4, 6].

Declaration of Competing Interest

None.

Acknowledgements

We would like to thank Professor K. Rauf, for his useful suggestions and encouragement. We sincerely thank our supervisor Dr. S.O. Makanjuola.

References

[1] D.S. Malik, N.J. Mordeson, M.K. Sen, Introduction to Abstract Algebra, Scientific Word Publishers, United States of America, 2007.
[2] J.M. Howie, Monograph on Semigroup of Mappings, School of Mathematics and Statistics, University of Saint Andrews, North Haught, United Kingdom, 2006.
[3] P.B. Bhattacharya, S.K. Jaini, S.R. Nagpaul, First Course in Linear Algebra, New Age International Publishers, London, New-Delhi, Nairobi, 2018.
[4] A. Umar, Some combinatorial problems in the theory of partial transformation semigroups, in: Proceedings of the 5th NBSAN Meeting, University of Saint Andrews, 2010.
[5] I.H. Sheth, Abstract Algebra, 2nd edition, Phi-Learning Private Limited, New Delhi, 2009 110001.
[6] O. Ganyushkin, V. Mazorchuk, An Introduction to Classical Finite Transformation Semigroup, Springer-Verlag, London, 2009.
[7] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.
[8] A.H. Cliiford, G.B. Preston, The Algebraic Theory of Semigroups, 2, Mathematical Surveys of the American Mathematical Society, 1967.
[9] H. Evis, V.C. Newson, An Introduction to Foundation and Fundamentals Concept of Mathematics, Holt, Reint, Winson, 1997.
[10] J.R.;. Joseph, A First Course in Abstract Algebra, Prentice-Hall, Upper Saddle River, New Jersey, 2005.
[11] A.G. Kurosh, The Theory of Groups, Chelsea, London, 1953.
[12] H. Tero, Lecture Notes on Semigroups, Department of Mathematics, University of Turku, Turku, Finland, 1996 FIN-20014.
[13] J.C. Alan, *Nine Chapters on the Semigroup Arts*. Lecture Notes for a Tour Through Semigroups. Creative Commons Attributions-Non Commercial-No Derivs 4.0, International Licence. Parto and Lisbon, 2009.