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*Research Article*

# Perturbed Galerkin Method for Solving Integro-Differential Equations

**K. Issa**,[1] **J. Biazar**,[2] **T. O. Agboola**,[1] **and T. Aliu**[1]

[1]*Department of Mathematics and Statistics, Kwara State University, Malete, Kwara State, Nigeria*
[2]*Department of Applied Mathematics, University of Guilan, Rasht, Iran*

Correspondence should be addressed to K. Issa; issakazeem@yahoo.com

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In this paper, perturbed Galerkin method is proposed to find numerical solution of an integro-differential equations using fourth kind shifted Chebyshev polynomials as basis functions which transform the integro-differential equation into a system of linear equations. The systems of linear equations are then solved to obtain the approximate solution. Examples to justify the effectiveness and accuracy of the method are presented and their numerical results are compared with Galerkin's method, Taylor's series method, and Tau's method which provide validation for the proposed approach. The errors obtained justify the effectiveness and accuracy of the method.

## 1. Introduction

Many researchers in the recent times have been concentrating their attention on the solution of integro-differential equation due to its application in many branches of Mathematical sciences such as Mathematical modeling (in biological application, engineering, and physical problems), Financial mathematics, control theory, and many of the problems are difficult to solve analytically. Different methods have been employed by many researchers to find the approximate solution of integro-differential equations such as Biazar and Salehi [1, 2] who applied the Chebyshev Galerkin method to study integro-differential equations of second kind, Jangveladze et al. [3] proposed finite element method for nonlinear integro-differential model, Fathy [4] proposed the Legendre-Galerkin method for solving linear the Fredholm integro-differential equations, Hosseini and Shahmorad [5–7] implemented the Tau method to obtain approximate solution of Fredholm-Volterra integro-differential equations with arbitrary polynomial bases and its corresponding error estimation. Homotopy perturbation methods were used in [8–10] to solve integro-differential equations, Chen et al. [11] studied the convergence and stability of Galerkin's method in their paper entitled "A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation". The Pell-Lucas matrix-collocation method was proposed by Demir et al. [12] to solve a class of Fredholm-type delay integro-differential equations with variable delays, convergence of a block-by-block method was proposed by Makroglou [13] for the solution of nonlinear Volterra integro-differential equations. Khater et al. [14] implemented Legendre polynomials to solve integral and integro-differential equations, Chang [15] implemented the extrapolation methods to obtain numerical solution of integro-differential equations, variational iteration method was used by Bildik [16] to obtain the approximate solution of general linear Fredholm integro-differential equations, Shahsavara [17] employed the Haar wavelets

to solve the linear Volterra and Fredholm integro-differential equations, Yalcinbas [18–20] used the Taylor polynomials to obtain approximate solution of high-order linear Volterra-Fredholm integro-differential equations, Golbabai and Seifollahi [21] implemented radial basis function networks to obtain numerical solution of linear integro-differential equations, Gümgüm [22] employed the Lucas polynomial together with standard and the Chebyshev-Lobatto collocation points to solve functional integro-differential equations involving variable delays, Šmarda and Khan [23] solved analytical properties and asymptotic behaviour of solutions for system of integro-differential equations and also in another paper they used an approach which combines topological method of Wazewski and Schauder's fixed point theorem to solve singular initial value problem for a system of integro-differential equations unsolved with respect to the derivative (see [24] for details).

The main motivation in this present work is proposed perturbed Galerkin approach to find approximate solution for integro-differential equations. The approximate solutions obtain by the proposed approach are more accurate and effective compared to the numerical solutions obtained by the Tau method and Galerkin method (GM).

## 2. Review of Some Orthogonal Polynomials

A sequence of real functions $\psi_r(z), r = 0, 1, \cdots$ is said to be orthogonal with respect to the weight function $\omega(z)$ over $[a, b]$, if

$$\langle \psi_m(z), \psi_n(z) \rangle = \int_a^b \psi_m(z)\psi_n(z)\omega(z)dz = \begin{cases} 0, m \neq n \\ \lambda_n, m = n \end{cases}, \tag{1}$$

where

$$\psi_n(z) = \sum_{i=0}^{n} a_i z^i. \tag{2}$$

Some commonly used orthogonal polynomials $\psi_n(z)$ are as follows:

*Legendre polynomials* $P_m(z)$: An orthogonal polynomial with recurrence relation and weight function:

$$P_{m+1}(z) = \frac{2m+1}{m+1} zP_m(z) - \frac{m}{m+1} P_{m-1}(z),$$

$$m \geq 1, \text{and } \omega(z) = 1, z \in [-1, 1], \tag{3}$$

respectively, where $P_0(z) = 1, P_1(z) = z$.

Table 1: Maximum absolute errors for Example 1 at different degrees of approximation $m$.

| $m$ | PM | GM [2] | CGM [1] |
|---|---|---|---|
| 4 | $8.08 \times 10^{-4}$ | $4.63 \times 10^{-3}$ | $1.91 \times 10^{-4}$ |
| 8 | $5.79 \times 10^{-7}$ | $5.61 \times 10^{-6}$ | $1.57 \times 10^{-7}$ |
| 12 | $2.93 \times 10^{-10}$ | $5.93 \times 10^{-9}$ | $1.88 \times 10^{-10}$ |

*Shifted Gegenbauer polynomials* $C_m^{(\alpha)}(z)$: An orthogonal polynomial in the interval $[a, b]$ having recurrence relation:

$$C_m^{(\alpha)}(z) = \frac{1}{m}\left[2(m-1+\alpha)\left(\frac{2z-(a+b)}{b-a}\right)C_{m-1}^{(\alpha)}(z) - (m-2+2\alpha)C_{m-2}^{(\alpha)}(z)\right],$$

$$m \geq 2, \text{where} C_m^{(\alpha)} = 1, C_1^{(\alpha)} = 2\alpha\left(\frac{2z-(a+b)}{b-a}\right), \tag{4}$$

with respect to the weight function $\omega(z) = [1 - (2z - (a+b)/b-a)^2]^{\alpha-1/2}$ (see [25–27] for more details).

*Jacobi polynomials*: The well-established Jacobi polynomials $P_m^{(\alpha,\gamma)}(z), m = 0, 1, \cdots$, with parameters $\alpha, \gamma > -1$ reference to the weight function $\omega(z) = (1-z)^{\alpha}(1+z)^{\gamma}$. The explicit forms of Jacobi polynomials are used in [28] as

$$P_m^{(\alpha,\gamma)}(z) = \frac{\Gamma(\alpha+m+1)}{m!\Gamma(\alpha+\gamma+m+1)} \sum_{n=0}^{m} \frac{m!\Gamma(\alpha+\gamma+n+m+1)}{(m-n)!n!\Gamma(\alpha+n+1)} \left(\frac{z-1}{2}\right)^n. \tag{5}$$

Equation (5) becomes the Legendre polynomial whenever $\alpha = \gamma = 0$, it reduces to the Chebyshev polynomial of the second kind when $\alpha = \gamma = 1/2$, while it gives the Gegenbauer polynomial of the form $C_m^{3/2}(z)$ whenever $\alpha = \gamma = 1$.

*Chebyshev polynomials of* $\nu$ *kinds*: Chebyshev polynomials are of different kinds $\nu$ with respect to their respective weight functions $\omega(z)$ defined here in the interval $[-1, 1]$ are

$$\psi_{\nu,m} = \begin{cases} T_m = \cos(m\phi), \nu = 1, \\ U_m = \dfrac{\sin(m+1)\phi}{\sin(\phi)}, \nu = 2, \\ V_m = \dfrac{\cos(m+1/2)\phi}{\cos(\phi/2)}, \nu = 3, \\ W_m = \dfrac{\sin(m+1/2)\phi}{\sin(\phi/2)}, \nu = 4, \end{cases} \tag{6}$$

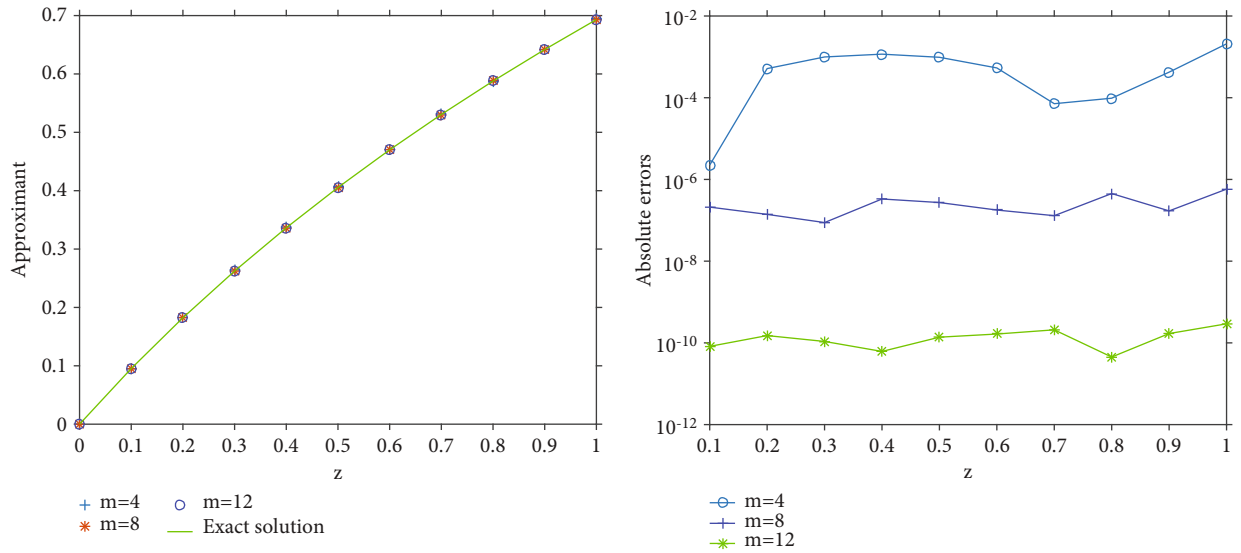



FIGURE 1: Approximate solutions and its corresponding absolute errors at different degrees of approximation $m$.

and their corresponding weight functions $\omega(z)$ are

$$\omega(z)_\nu = \begin{cases} \left(1-z^2\right)^{-\frac{1}{2}}, \nu = 1, \\ \left(1-z^2\right)^{\frac{1}{2}}, \nu = 2, \\ \left(\dfrac{1+z}{1-z}\right)^{\frac{1}{2}}, \nu = 3, \\ \left(\dfrac{1-z}{1+z}\right)^{\frac{1}{2}}, \nu = 4. \end{cases} \tag{7}$$

The Chebyshev polynomials of all kinds are said to be orthogonal reference to their respective kind $\nu$ in the interval $[-1, 1]$, if

$$\begin{aligned} &\langle \psi_{\nu,m}(z), \psi_{\nu,n}(z) \rangle \\ &= \begin{cases} \langle \psi_{\nu,m}(z), \psi_{\nu,n}(z) \rangle = \displaystyle\int_{-1}^{1} \omega(z)_\nu \psi_{\nu,m}(z)\psi_{\nu,n}(z)dz = \begin{cases} 0, m \neq n \\ \dfrac{\pi}{2}, m = n \end{cases}, \nu = 1, 2, \\ \langle V_m(z), V_n(z) \rangle = \displaystyle\int_{-1}^{1} \omega(z)_\nu V_m(z)V_n(z)dz = \begin{cases} 0, m \neq n \\ \pi, m = n \end{cases}, \nu = 3, \\ \langle W_m(z), W_n(z) \rangle = \displaystyle\int_{-1}^{1} \omega(z) W_m(z)W_n(z)dz = \begin{cases} 0, m \neq n \\ \pi, m = n \end{cases}, \nu = 4, \end{cases} \end{aligned} \tag{8}$$

see details in [1, 29].

TABLE 2: Maximum absolute errors for Example 2 at different degrees of approximation $m$.

| $m$ | PM | GM [2] | CGM [1] |
|---|---|---|---|
| 4 | $8.12 \times 10^{-4}$ | $2.30 \times 10^{-3}$ | $4.19 \times 10^{-4}$ |
| 8 | $7.49 \times 10^{-10}$ | $1.38 \times 10^{-8}$ | $4.19 \times 10^{-10}$ |
| 12 | $9.40 \times 10^{-16}$ | $8.09 \times 10^{-15}$ | $1.20 \times 10^{-16}$ |

## 3. Description of Perturbed Galerkin Method

In this work, we consider integro-differential equations of the form:

$$\sum_{i=0}^{n} s_i(z) \frac{d^i}{dz^i}\varphi(z) = g(z) + \beta_1 \int_a^b \rho_1(z,t)\varphi(t)dt + \beta_2 \int_a^z \rho_2(z,t)\varphi(t)dt, \tag{9}$$

subject to the initial condition:

$$\varphi^i(a) = \tau_i, i = 0, 1, \cdots, (n-1), \tag{10}$$

where $\varphi(z)$ is an unknown function, $s_i(z)$, $g(z)$, $\rho_1(z,t)$, and $\rho_2(z,t)$ are known functions, $n$ is the order of Equation (9), $\beta_1, \beta_2$ are real numbers and $z$ is the independent variable of the functions which appear throughout this paper, defined in the interval $[a, b]$ except stated otherwise.

Suppose $\varphi_m(z)$ is the approximate solution of degree $m$ to the function $\varphi(z)$, then we write

$$\varphi_m(z) = \sum_{j=0}^{m} \alpha_j W_j\left(\frac{2z-(a+b)}{b-a}\right), \tag{11}$$

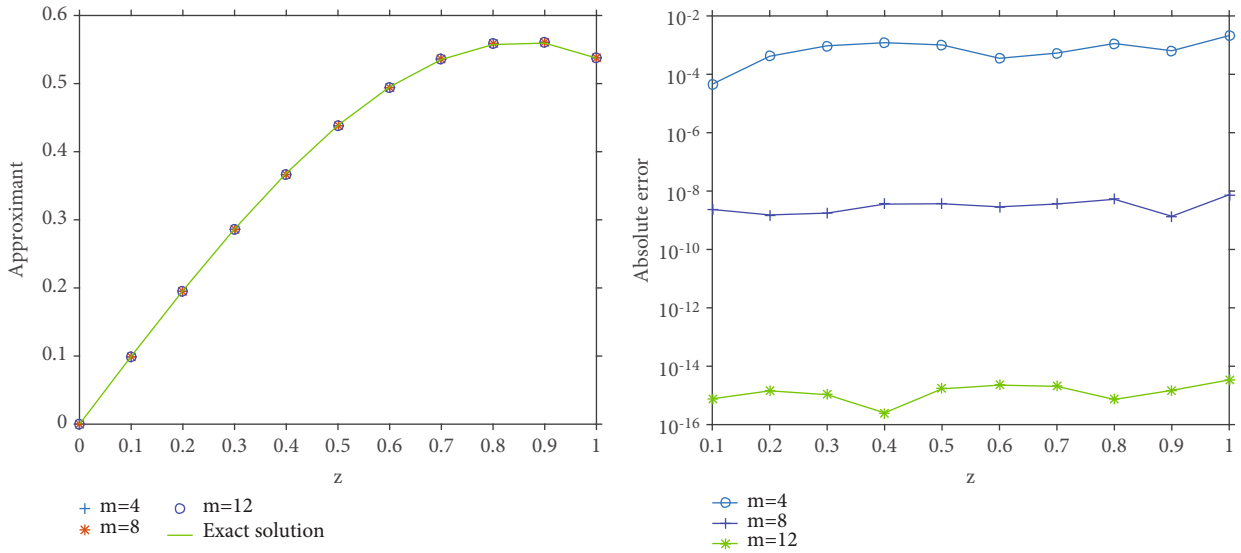



Figure 2: Approximate solution for Example 2 and its absolute errors.

where $\alpha_j$ are unknowns to be determined and $W_j, j = 0, 1, \cdots, m$ are fourth kind shifted Chebyshev polynomials.

Adding perturbation terms $H_m(z)$ to the right-hand side of approximate form of Equation (9), we obtain

$$\sum_{i=0}^{n} s_i(z)\frac{d^i}{dz^i}\varphi_m(z) - H_m(z) - \beta_1\int_a^b \rho_1(z,t)\varphi_m(t)dt - \beta_2\int_a^z \rho_2(z,t)\varphi_m(t)dt = g(z), \tag{12}$$

then multiply both sides of Equation (12) by $W_k(2z-(a+b)/b-a), k = n, n+1, n+m+1$ and integrate the resulting equation with respect to the independent variable, over $[a, b]$, we obtain

$$\begin{aligned}\int_a^b \Bigg[\sum_{i=0}^{n} s_i(z)\sum_{j=0}^{m}\left(\frac{2}{b-a}\right)^i \alpha_j \frac{d^i}{dz^i} W_j\left(\frac{2z-(a+b)}{b-a}\right) &- \beta_1\int_a^b \rho_1(z,t)\sum_{j=0}^{m}\alpha_j W_j\left(\frac{2t-(a+b)}{b-a}\right)dt \\ &- \beta_2\int_a^z \rho_2(z,t)\sum_{j=0}^{m}\alpha_j W_j\left(\frac{2t-(a+b)}{b-a}\right)dt \\ &- \sum_{r=0}^{n}\eta_{r+1} W_{m-n+r+1}\left(\frac{2z-(a+b)}{b-a}\right)\Bigg] W_k\left(\frac{2z-(a+b)}{b-a}\right)dz \\ = \int_a^b \left[g(z) W_k\left(\frac{2z-(a+b)}{b-a}\right)\right]dz, k = n, n+1, \cdots, n+m+1,\end{aligned} \tag{13}$$

where

$$H_m(z) = \sum_{r=0}^{n}\eta_{r+1} W_{m-n+r+1}\left(\frac{2z-(a+b)}{b-a}\right), \tag{14}$$

$m$ is the degree of approximation.

Table 3: Maximum absolute errors for Example 3 at different degree of approximation $m$.

| $m$ | PM | GM [2] | CGM [1] |
|---|---|---|---|
| 4 | $7.38\times10^{-5}$ | $7.20\times10^{-3}$ | $8.14\times10^{-5}$ |
| 8 | $8.59\times10^{-11}$ | $2.31\times10^{-9}$ | $8.61\times10^{-11}$ |
| 12 | $8.91\times10^{-17}$ | $9.24\times10^{-16}$ | $1.67\times10^{-17}$ |

Putting Equation (13) in matrix form, we have

$$\chi\alpha = G, \tag{15}$$

where $\chi$ is a matrix of $(m+2)\times(m+n+2)$, $\alpha$ and $G$ are column matrices of $(m+2)\times 1$. We derived the remaining equations from the initial conditions (10), that is

$$\begin{aligned}\frac{d^i}{dz^i}\varphi_m(a) &= \sum_{i=0}^{m}\left(\frac{2}{b-a}\right)^i \alpha_i \frac{d^i}{dz^i} W_i\left(\frac{2z-(a+b)}{b-a}\right)\Bigg|_{z=a} \\ &= \tau_i, i = 0, 1, n-1.\end{aligned} \tag{16}$$

Solving Eqs. (13) and (16) simultaneously, we obtain the values of the unknowns $\alpha_j, j = 0, 1, \cdots, m$, we then substitute in Equation (11) to obtain the approximate solution of degree $m$.

## 4. Numerical Experiments

In this section, we illustrate the method discussed (present method (PM)) in Section 3 to solve some selected examples from the literature [1, 2, 7, 21]. To achieve the desired results, we used Maple for all the computation and MATLAB for graphical interpretation of the results. The results are compared with the results obtained using radial basis function (RBF) [21], the Galerkin method (GM) [2], the Chebyshev-Galerkin method (CGM) of first kind [1],

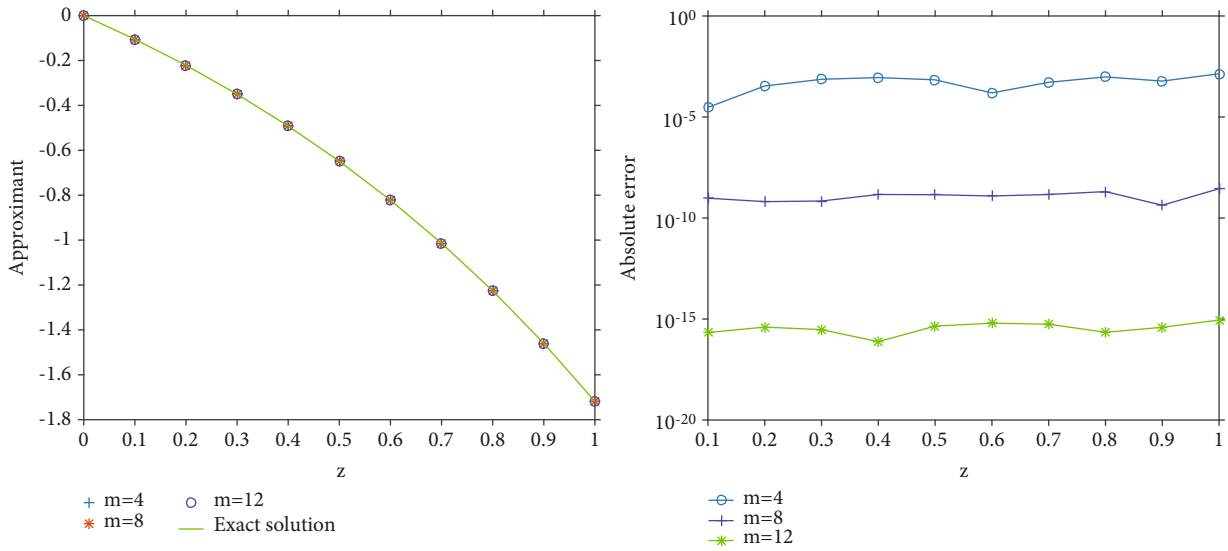



Figure 3: Approximate solutions and its absolute errors.

and Tau method [7]. We compute the absolute maximum error $\xi_m$ and compare with the existing results from the literature, written as below:

$$\xi_m = \max_{0\le i\le 100} |\varphi(z_i) - \varphi_m(z_i)|, z_i = a + ih. \tag{17}$$

*Example 1.* Considering the Fredholm integro-differential equation in [1, 2, 21],

$$\varphi'(z) = \varphi(z) - \frac{1}{2}z + \frac{1}{1+z} - \log(1+z) + \frac{1}{(\log(2))^2}\int_0^1 \frac{z}{1+t}\varphi(t)dt, 0 \le z \le 1 \tag{18}$$

subject to initial condition $\varphi(0) = 0$.

The exact solution $\varphi(z) = \log(1+z)$.

Replacing $\varphi(z)$ in Equation (18) with the corresponding approximate solution defined in Equation (11) and add the perturbation terms defined in Equation (14), we obtain

$$\varphi'_m(z) - \varphi_m(z) - \frac{1}{(\log(2))^2}\int_0^1 \frac{z}{1+t}\varphi_m(t)dt - \sum_{r=0}^{n}\eta_{r+1}W_{m-n+r+1}(2z-1) = \frac{1}{1+z} - \frac{1}{2}z - \log(1+z), 0 \le z \le 1. \tag{19}$$

Seeking an approximate solutions of degrees 4,8, and 12 (that is $m = 4, 8$, and 12), we multiply both sides of Equation (19) by $W_k(2z-1), k = 1, 2, \cdots, m+2$ (since $k = n, n+1, \cdots, n+m+1$, as explained in Equation (13)) and then integrate the resulting equation in the interval $[0, 1]$, we obtain

$$\int_0^1 \left[\sum_{j=0}^{m} 2^j\alpha_j \frac{d}{dz}W_j(2z-1) - \sum_{j=0}^{m}\alpha_j W_j(2z-1) - \frac{1}{(\log(2))^2}\int_0^1 \frac{z}{1+t}\sum_{j=0}^{m}\alpha_j W_j(2t-1)dt - \sum_{r=0}^{1}\eta_{r+1}W_{r+4}(2z-1)\right] W_k(2z-1)dz = \int_0^1 \left[\frac{1}{1+z} - \frac{1}{2}z - \log(1+z)\right] W_k(2z-1)dz, k = 1, 2, \cdots, m+2. \tag{20}$$

From Equation (20) we obtain $m+2$ equations with $m+3$ unknowns, to obtain a unique solutions, we use initial condition $\varphi(0) = 0 \cong \varphi_m(0)$, that is

$$\alpha_0 - \alpha_1 + \cdots + (-1)^m\alpha_m = 0. \tag{21}$$

We then solve Equations (20) and (21) to obtain the approximate solutions.

Table 1 shows the maximum absolute errors at different degrees of approximation. Figure 1 shows the exact and its corresponding approximate solutions at different degrees of approximation $m$.

Table 4: Absolute maximum errors for Example 4 at different degrees of approximation $m$.

| $m$ | PM | GM [2] | CGM [1] | Tau method [7] |
|---|---|---|---|---|
| 5 | $4.66 \times 10^{-4}$ | $3.19 \times 10^{-3}$ | $7.12 \times 10^{-4}$ | $3.19 \times 10^{-3}$ |
| 10 | $5.49 \times 10^{-10}$ | $2.10 \times 10^{-6}$ | $1.00 \times 10^{-9}$ | $2.10 \times 10^{-6}$ |
| 15 | $1.27 \times 10^{-17}$ | $7.50 \times 10^{-11}$ | $1.44 \times 10^{-17}$ | $7.50 \times 10^{-11}$ |

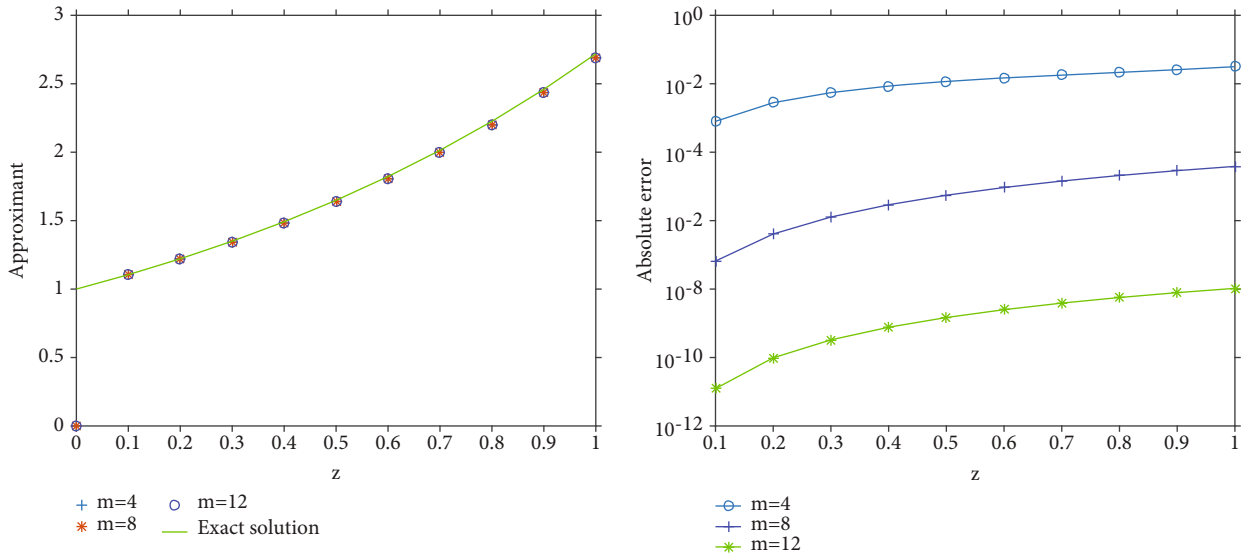



Figure 4: The approximate solution for Example 4 and its absolute errors at different values of $m$.

*Example 2.* Consider the Volterra integro-differential equation in [1, 2, 30]

$$\varphi'(z) - \int_0^z \varphi(t)dt = 1 - 2z\sin(z); 0 \le z \le 1, \varphi(0) = 0, \quad (22)$$

with analytic solution $\varphi(z) = z\cos(z)$.

Table 2 displays the comparison of the absolute maximum errors at different degrees of approximation relative to the existing ones from the literature. Figure 2 exhibits the exact solution and the corresponding approximate solutions.

*Example 3.* Consider the following Volterra integro-differential equation [1, 2, 30]

$$\varphi'(z) + \int_0^z t\varphi(t)dt = -1 + \frac{1}{2}z^2 - z\exp(z); 0 \le z \le 1, \varphi(0) = 0, \quad (23)$$

the exact solution is $\varphi(z) = 1 - \exp(z)$.

Absolute maximum errors obtained are tabulated in Table 3. Figure 3 presents the exact solution and the corresponding approximate solutions at different degrees of approximation $m$.

*Example 4.* Consider the Fredholm-Volterra integro-differential equation ([1, 2, 7, 18])

$$\varphi''(z) + z\varphi'(z) - z\varphi(z) = e^z - 2\sin(z) + \int_{-1}^{1} \sin(z)e^t\varphi(t)dt, -1 \le z \le 1, \quad (24)$$

subject to initial conditions $\varphi(0) = 1, \varphi'(0) = 1$ and with the exact solution given as $y(x) = \exp(z)$.

The absolute maximum errors are presented in Table 4 together with the results from the literature with Figure 4 showing the exact and the corresponding approximate solutions.

## 5. Discussion of Results and Conclusion

*5.1. Discussion of Results.* Tables 1–4 show the absolute maximum errors obtained for Examples 1–4, as compared with the existing results in the literature. It was observed from the results obtained that the present method gives a better accuracy compared to the results obtained in [2, 7, 21], while it produced the same degree of accuracy when compare with CGM [1]. Figures 1–4 display the exact and its corresponding approximate solutions at different values of $m$, it is obvious from the figures that, as the degree of approximation increases the accuracy improves, that is, the maximum absolute errors decline.

*5.2. Conclusion.* In this work, we have proposed the perturbed Galerkin method for solving integro-differential equations using fourth kind shifted Chebyshev polynomials as basis functions. The Galerkin approach was used to transform the equation into a system of algebraic equations with the resulting equations solved to obtain the coefficients of the approximate solution of degree $m$. The method was implemented on some selected problems from the literature. The results have been compared with the GM, CGM, and Tau method solutions. The present approach gives a better accuracy compared to the results obtained via GM and Tau method while it is in good agreement with the results obtained via CGM. Sequel to the results, it becomes obvious that the proposed method is effective and accurate. In conclusion, the perturbed Galerkin approach can be considered

a nice and alternative method to the existing numerical techniques.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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