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AN ANALOGUE OF THE TAU METHOD FOR ORDINARY DIFFERENTIAL EQUATION

KAZEEM ISSA and **R. B. ADENIYI**
Department of Mathematics

Abstract : In this paper, we construct three orthogonal polynomials which are incorporated into the perturbation term of a numerical scheme analogous to the Ortiz recursive formulation of the Lanczos tau method. The method was implemented on some selected problems and the accuracy obtained justifies the desirability of the numerical scheme.

Introduction

The tau method of Lariczos, first proposed 1938 solves the linear ODE:

$$Ly(x) \equiv \sum_{r=0}^{Nr} Pr_k(x) y^{(k)}(x) = \sum_{r=0}^{\sigma} f_r x^r, \quad a \le x \le b \tag{1.1a}$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \quad k = 1(1)m \tag{1.1b}$$

by seeking an approximant:

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad r < +\infty \tag{1.3}$$

of y(x) which is the exact solution of the corresponding perturbed system:

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \tag{1.4a}$$

$$L^* y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \tag{1.4b}$$

$p_r(x)$ is an r-th degree polynomial function; $y^k(x)$ is the derivative of order k of y(x); f_r, α_k, α_{rk}, x_{rk}, a and b are all real constants, τ_{m+s-r} is a tau parameter to be determined and $T_n(x)$ is the n-th degree chebyshev polynomials:

$$T_n(x) = \cos[n \cos^{-1}(2x - a - b)/(b - a) - 1] \equiv \sum_{k=0}^{n} C_k^{(n)} x^k, \quad a \le x \le b \tag{1.5}$$

The number s, called the overdetermination number of the (1.1a), is given by

$$s = \max\{|N_r - r|, \quad 0 \le r \le m\} \tag{1.6}$$

In 1956, Lanczos introduced the use of canonical polynomials $\{Q_r(x)\}, r = 0,1,2\ldots$ into the

tau method. The difficulty involved in the construction of these polynomials was removed by Ortiz in 1969 when he reported a recursive generation of the polynomials, $Q_r(x)$, defined by

$$LQ_r(x) = x^r \tag{1.7}$$

In the real section, we shall construct a set of three orthogonal polynomials as replacement to the chebyshev polynomials above and, which when adopted in (1.4), produces results which compare favourably with that obtained from (1.4), Section 3 focuses on the present work while section 4 presnts some numerical examples for implementation with the proposed sheme. Finall, we conclude this paper with some remarks in section 5.

2 Construction of Orthogonal Polynomials

Orthogonal polynomials have been widely used in problems involving the approximation of functions such as in the econornization of power series, mini-max approximation, Gaussian quadrature techniques, solution of both integral and differential equation as well as in collacation techniques, among others.

The construction of these polynomials may be based on the three equations of Gram Schmidt orthogonalization principle namel :

$$\varphi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \tag{2.1}$$

$$\varphi_n(1) = 1 \tag{2.2}$$

$$\langle \varphi_n, \varphi_m \rangle = 0 \equiv \int_a^b \omega(x)\varphi_n(x)\varphi_m(x)dx \tag{2.3}$$

where $\{\varphi_n(x)\}$, $a \le x \le b$ is the orthogonal system in question.

By employing equations (2.1) – (2.3) for three cases of $\omega(x)$ and [a, b], we obtained the following results:

2.1 Case $\omega(x) \equiv 1 - x, [a,b] \equiv [-1,1]$

For this case we obtained

$$\varphi_0(x) = 1$$

$$\varphi_1(x) = \frac{1}{4}(1 + 3x)$$

$$\varphi_2(x) = \frac{1}{6}(-1 + 2x + bx^2)$$

$$\varphi_3(x) = \frac{1}{32}(-3 - 15x + 15x^2 + 35x^3)$$

$$\varphi_4(x) = \frac{1}{40}(3 - 12x - 42x^2 + 28x^3 + 63x^4)$$

$$\varphi_5(x) = \frac{1}{96}(5 + 35x - 70x^2 - 210x^3 + 105x^4 + 231x^5) \tag{2.4}$$

$$\varphi_6(x) = \frac{1}{112}(-5 + 30x + 135x^2 - 180x^3 - 495x^4 + 198x^5 + 429x^6)$$

$$\varphi_7(x) = \frac{1}{1024}(-35 - 315x + 945x^2 + 3465x^3 - 3465x^4 - 9009x^5 + 3003x^6 + 6435x^7)$$

$$\varphi_8(x)=\frac{1}{1152}(35-280x-1540x^2+3080x^3+10010x^4-8008x^5-20020x^6+5720x^7+12155x^8)$$

$$\varphi_9(x)=\frac{1}{2560}(63+693x-2772x^2-12012x^3+18018x^4+54054x^5-36036x^6-87516x^7+21879x^8+46189x^9)$$

2.2 Case $\omega(x)\equiv 1+x,[a,b]\equiv[-1,1]$:

$$\varphi_0(x)=1$$

$$\varphi_1(x)=\frac{1}{2}(-1+3x)$$

$$\varphi_2(x)=\frac{1}{2}(-1-2x+5x^2)$$

$$\varphi_3(x)=\frac{1}{8}(3-15x+15x^2+35x^3)$$

$$\varphi_4(x)=\frac{1}{8}(3+12x-42x^2-28x^3+63x^4)$$

$$\varphi_5(x)=\frac{1}{16}(-5+35x+70x^2-210x^3-105x^4+231x^5) \qquad (2.5)$$

$$\varphi_6(x)=\frac{1}{16}(-5-30x+135x^2+180x^3-495x^4-198x^5+429x^6)$$

$$\varphi_7(x)=\frac{1}{128}(35-315x-945x^2+3465x^3+3465x^4-9009x^5-3003x^6+6435x^7)$$

$$\varphi_8(x)=\frac{1}{128}(35+280x-1540x^2-3080x^3+10010x^4+8008x^5-20020x^6-5720x^7+12155x^8)$$

$$\varphi_9(x)=\frac{1}{256}(-63+693x+2772x^2-12012x^3-18018x^4+54054x^5+36036x^6-87516x^7-21879x^8+46189x^9)$$

2.3 Case $\omega(x)\equiv 1-x^2,[a.b]\equiv[-1,1]$:

$$\varphi_0(x)=1$$

$$\varphi_1(x)=x$$

$$\varphi_2(x)=\frac{1}{4}(1+5x^2)$$

$$\varphi_3(x)=\frac{1}{4}(-3x+7x^3)$$

$$\varphi_4(x)=\frac{1}{8}(1-14x^2+21x^4)$$

$$\varphi_5(x)=\frac{1}{8}(5x-30x^3+33x^5) \qquad (2.6)$$

$$\varphi_6(x)=\frac{1}{64}(-5+135x^2-495x^4+429x^6)$$

$$\varphi_7(x)=\frac{1}{64}(-35x+385x^3-1001x^5+715x^7)$$

$$\varphi_8(x) = \frac{1}{128}(7 + \cdot 308x^2 + 2002x^4 - 4004x^6 + 2431x^8)$$

$$\varphi_9(x) = \frac{1}{128}(63x - 1092x^3 + 4914x^5 - 7956x^7 + 4199x^9)$$

We remark here that the polynomials of even order are even functions of x and those of odd order are odd functions of x for the polynomials corresponding to weight function $\omega(x) = 1 - x^2$, and this property is also exhibited by the chebyshev polynomials because both belong to the same functions called ultra-spherical polynomials.

For purpose of adopting these sets of polynomials in the interval [0,1], we used the transformation

$$U = \frac{1}{2}(b-a)x + \frac{1}{2}(a+b), \quad -1 \le x \le 1$$

i.e

$$u = \frac{1}{2}(x+1) \text{ or } x = 2u - 1 \tag{2.7}$$

It is convienent to have a special notation for this range, and following Lanczos we write

$$\varphi_r^*(x) = \varphi_r(2x-1), \quad 0 \le x \le 1$$

All the properties of $\varphi_r^*(x)$ can be deduced from those of $\varphi_r(2x-1)$ and in particular, corrsponding to (2.4), (2.5), (2.6) with weight functions $(1 - x)$, $(1 + x)$ and $(1 - x^2)$, respectively, we can find formulae of the kind

2.4 Case $\omega(x) = 1 - x, [0.1]$:

$$\varphi_0^*(x) = 1$$

$$\varphi_1^*(x) = \frac{1}{2}(-1 + 3x)$$

$$\varphi_2^*(x) = \frac{1}{3}(1 - 8x + 10x^2)$$

$$\varphi_3^*(x) = \frac{1}{4}(-1 + 15x - 45x^2 + 35x^3)$$

$$\varphi_4^*(x) = \frac{1}{5}(1 - 24x + 126x^2 - 224x^3 + 126x^4)$$

$$\varphi_5^*(x) = \frac{1}{6}(-1 + 35x - 280x^2 + 840x^3 - 105x^4 + 462x^5) \tag{2.8}$$

$$\varphi_6^*(x) = \frac{1}{7}(1 - 48x + 540x^2 - 2400x^3 + 4950x^4 - 4752x^5 + 1716x^6)$$

$$\varphi_7^*(x) = \frac{1}{8}(-1 + 63x - 945x^2 + 5775x^3 - 17325x^4 + 27027x^5 - 21021x^6 + 6435x^7)$$

$$\varphi_8^*(x) = \frac{1}{9}(1 - 80x + 1540x^2 - 12320x^3 + 50050x^4 - 112112x^5 + 140140x^6 - 91520x^7 + 24310x^8)$$

$$\varphi_9^*(x) = \frac{1}{10}(-1 + 99x - 2376x^2 + 24024x^3 - 126126x^4 + 378378x^5 - 672672x^6 + 700128x^7 - 393822x^8 + 92378x^9)$$

2.5 Case $\omega(x) = 1 + x, [0,1]$:

$$\varphi_0^*(x) = 1$$
$$\varphi_1^*(x) = (-2 + 3x)$$
$$\varphi_2^*(x) = (3 - 12x + 10x^2)$$
$$\varphi_3^*(x) = (-4 + 30x - 60x^2 + 35x^3)$$
$$\varphi_4^*(x) = (5 - 60x + 210x^2 - 280x^3 + 126x^4)$$
$$\varphi_5^*(x) = (-6 + 105x - 560x^2 + 1260x^3 - 1260x^4 + 462x^5) \qquad (2.9)$$
$$\varphi_6^*(x) = (7 - 168x + 1260x^2 - 4200x^3 + 6930x^4 - 5544x^5 + 1716x^6)$$
$$\varphi_7^*(x) = (-8 + 252x - 2520x^2 + 11550x^3 - 27720x^4 + 36036x^5 - 24024x^6 + 6435x^7)$$
$$\varphi_8^*(x) = (9 - 360x + 4620x^2 - 27720x^3 + 90090x^4 - 168168x^5 + 180180x^6 - 102960x^7 + 24310x^8)$$
$$\varphi_9^*(x) = (-10 + 495x - 7920x^2 + 60060x^3 - 252252x^4 + 630630x^5 - 960960x^6 + 875160x^7 - 437580x^8 + 92378x^9)$$

2.6 Case $\omega(x) = 1 - x^2, [0,1]$:

$$\varphi_0^*(x) = 1$$
$$\varphi_1^*(x) = (-1 + 2x)$$
$$\varphi_2^*(x) = (1 - 5x + 5x^2)$$
$$\varphi_3^*(x) = (-1 + 9x - 21x^2 + 14x^3)$$
$$\varphi_4^*(x) = (1 - 14x + 56x^2 - 84x^3 + 42x^4)$$
$$\varphi_5^*(x) = (-1 + 20x - 120x^2 + 300x^3 - 330x^4 + 132x^5) \qquad (2.10)$$
$$\varphi_6^*(x) = (1 - 27x + 225x^2 - 825x^3 + 1485x^4 - 1287x^5 + 429x^6)$$
$$\varphi_7^*(x) = (-1 + 35x - 385x^2 + 1925x^3 - 5005x^4 + 7007x^5 - 5005x^6 + 1430x^7)$$
$$\varphi_8^*(x) = (1 - 44x + 616x^2 - 4004x^3 + 14014x^4 - 28028x^5 + 32032x^6 - 19448x^7 + 4862x^8)$$
$$\varphi_9^*(x) = (-1 + 54x - 936x^2 + 7644x^3 - 34398x^4 + 91728x^5 - 148512x^6 + 143208x^7 - 75582x^8 + 16796x^9)$$

3. An Analogue of the Lanczos method for ODE

If we replace $T_n(x)$ by $\varphi_n(x)$ the tau approximation process of Section 1, we then have the problem:

$$Ly_n(x) \equiv \sum_{r=0}^{m} \left(\sum_{k=0}^{Nr} p_r k^{x^k} \right) y_n^{(r)}(x) \qquad (3.1a)$$
$$= \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} v_{m+s-r} \varphi_{n-m+r+1}(x)$$
$$L^* y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \qquad (3.1b)$$

where $\varphi_n(x)$ is given by any of (2.8), (2.9) and (2.10), υ replaces τ but all the other parameters of (3.1) retain their definition as given in Section-1.

For the recursive formulation of this method, we introduce the canonical polynomials, $Q_r(x)$, given by (1.7) and then replace (1.3) by

$$y_n(x) = \sum_{r=0}^{n} a_r Q_r(x) \qquad (3.2)$$

as the appropriate approximant of y(x), the solution of (1.1). The justification for the canonical polynomials is derived from the following:

(i) canonical polynomials are independent of the interval in which the solution is sought, allowing for easy segmentation of the domain;

(ii) they are permanent, in the sense that if an approximation of higher degree is required, the computation does not need to be repeated from scratch; and

(iii) they are also independent of the supplementary conditions of the problem which can be initial boundary or multi-point conditions.

Now, from (3.la), i.e,

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} v_{m+s-r}\varphi_{n-m+r+1}(x)$$

we have

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} v_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} x^k \tag{3.3}$$

$$Ly_n(x) = \sum_{r=0}^{F} f_r LQ_r(x) + \sum_{r=0}^{m+s-1} v_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} LQ_k(x) \tag{3.4}$$

$$= L\left\{\sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} v_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} Q_k(x)\right\} \tag{3.5}$$

Also the assumption of the existence of L^{-1} given rise to the approximation

$$= \sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} v_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} Q_k(x) \tag{3.6}$$

to the desired solution y(x) of (1.1) In what now follow in the next section, we exhibit the above phenomenon for certain problems.

4. Numerical Examples

We consider two problems for implementation with the method in the above discourse

Example 4.1

$$2(1+x)y'(x) + y(x) = 0, \quad 0 \le x \le 1 \tag{i}$$

$$y(0) = 1 \tag{ii}$$

The analytic solution is $y(x) = (1+x)^{-\frac{1}{2}}$

In this case, $$Ly(x) = \left\{2(1+x)\frac{d}{dx} + 1\right\} y(x)$$

with $$L = 2(1+x)\frac{d}{dx} + 1 \tag{iii}$$

Then using (3.5) we have $$Ly_n(x) = v\varphi_n^*(x)$$

where $y_n(x)$ is given by (3.2) and v_1, v_2 fvre to be determined; $\varphi_n^*(x)$ can be any of the orthogonal polynomials of section 2. So,

$$Ly_n(x) = v\sum_{r=0}^{n} K_r^{(n)} x^r$$

$$= \nu \sum_{r=0}^{n} K_r^{(n)} L Q_r(x)$$

$$= L\left\{ \nu \sum_{r=0}^{n} K_r^{(n)} Q_r(x) \right\}$$

giving. $$y_n(x) = \nu \sum_{r=0}^{n} K_r^{(n)} Q_r(x) \qquad (iv)$$

By virtue of (3.2) this becomes $\sum_{r=0}^{n} a_r Q_r = \nu \sum_{r=0}^{n} K_r^{(n)} Q_r(x)$

This gives $a_n = \nu K_r^{(n)}$, $r = 0(1)n(\nu)$

Applying the condition (ii) in (iv) we have

$$\nu \sum_{r=0}^{n} K_r^{(n)} Q_r(0) = 1$$

That is, $$\nu = \frac{1}{\Sigma_{r=0}^{n} K_r^{(n)} Q_r(0)}$$

To obtain $Q_r(x)$. we have from (iii) and (1.7)

$$Lx^r = 2rx^{r-1} + (2r+1)x^r$$
$$= 2rQ_{r-1}(x) + (2r+1)Q_r(x)$$

$$Q_r(x) = \frac{1}{2r+1}(x^r + 2rQ_{r-1}(x)), \quad r = 0,1,..$$

This yields the members

$Q_0(x) = 1, Q_1(x) = \frac{1}{3}(x-2), Q_2(x) = \frac{1}{15}(3x^2 - 4x + 8), Q_3(x) = \frac{1}{35}(5x^3 - 6x^2 + 8x - 16),$
$Q_4(x) = \frac{1}{315}(35x^4 - 40x^3 + 48x^2 - 64x + 128), Q_5(x) = \frac{1}{693}(63x^5 - 70x^4 + 80x^3 - 96x^2 +$
$128x - 256), Q_6(x) = \frac{1}{3003}(231x^6 - 252x^5 + 280x^4 - 320x^3 + 384x^2 - 512x + 1024),$
$Q_7(x) = \frac{1}{6435}(429x^7 - 462x^6 + 504x^5 - 560x^4 + 640x^3 - 768x^2 + 1024x - 2048)$
$Q_0(0) = 1, Q_1(0) = \frac{-2}{3}, Q_2(0) = \frac{8}{15}, Q_3(0) = \frac{-16}{35}, Q_4(0) = \frac{128}{315}, Q_5(0) = \frac{-256}{693}, Q_6(0) = \frac{1024}{3003},$
$Q_7(0) = \frac{-2048}{6435}$

Let us seek a fifth degree approximant for which n = 5 and have, respectively, the equations

$$\nu = \frac{-3}{4900}$$

and $$\nu = \frac{-2}{385}$$

$$\nu = \frac{-3}{1195}$$

for the cases $\omega(x) = 1 + x$, $\omega(x) = 1 - x$ and $\omega(x) = 1 - x^2$

We solve (v) to get, respectively, the approximants:

$$y_5(x) = 1 - \frac{2441}{4900}x + \frac{438}{1225}x^2 - \frac{59}{245}x^3 + \frac{4}{35}x^4 - \frac{63}{2450}x^5$$

$$y_5(x) = 1 - \frac{1039}{2079}x + \frac{424}{1155}x^2 - \frac{184}{693}x^3 + \frac{14}{99}x^4 - \frac{126}{3465}x^5$$

$$y_5(x) = 1 - \frac{596}{1195}x + \frac{432}{1195}x^2 - \frac{60}{239}x^3 + \frac{30}{239}x^4 - \frac{36}{1195}x^5$$

By the tau method, we also obtained

$y_5(x) = 0.999999989 - 0.499659611x + 0.366235117x^2 - 0.259811446x^3 + 0.132027596x^4 - 0.0316866232x^5$

The results for this approximation of degree 5 are presented in Tables:

Table-4.1(a) : Numerical Results For Example 4.1($w(x) = 1+x$)

X_k	Approximate Solution	Exact Solution	Error
0.0	1.00000000	1.000000000	0.0
0.1	0.953529539	0.953462589	6.69495E-05
0.2	0.912917486	0.912870929	4.65565E-05
0.3	0.8770918	0.877058019	3.37807E-05
0.4	0.845193012	0.845154255	3.87575E-05
0.5	0.816543367	0.816496581	4.67864E-05
0.6	0.790615967	0.790569415	4.65523E-05
0.7	0.767003914	0.766964989	3.89254E-05
0.8	0.745389453	0.745355992	3.34606E-05
0.9	0.725513114	0.72547625	3.68642E-05
1.0	0.707142857	0.707106781	3.6076E-05

Table-4.1(b) : Numerical Results For Example 4.1 $w(x) = 1 - x$

X_k	Approximate Solution	Exact Solution	Error
0.0	1.000000000	1.000000000	0.0
0.1	0.953443311	0.953462589	1.9278E-05
0.2	0.912822611	0.912870929	4.83183E-05
0.3	0.876999371	0.877058019	5.86485E-05
0.4	0.845087184	0.845154255	6.70705E-05
0.5	0.816408129	0.816496581	8.8452E-05
0.6	0.790449131	0.790569415	1.20284E-4
0.7	0.76681833	0.766964989	1.46659E-4
0.8	0.745201439	0.745355992	1.54553E-4
0.9	0.725318113	0.72547625	1.58138E-4
1.0	0.706878307	0.707106781	2.28474E-4

Table-4.1(c) : Numerical Results For Example 4.1$w(x) = 1 - x^2$

X_k	Approximate Solution	Exact Solution	Error
0.0	1.000000000	1.00000000	0.0
0.1	0.953501791	0.953462589	3.92E-05
0.2	0.912894126	0.912870929	2.32E-05
0.3	0.877077423	0.877058019	1.94E-05
0.4	0.845181054	0.845154255	2.68E-05
0.5	0.816527197	0.816496581	3.06E-05
0.6	0.790594678	0.790569415	2.53E-05
0.7	0.766982828	0.766964989	1.78E-05
0.8	0.745375331	0.745355992	1.93E-05
0.9	0.725504067	0.72547625	2.78E-05
1.0	0.707112971	0.707106781	6.19E-06

Table-4.1(d) : Numerical Results For Example 4.1, Chebyshev

X_k	Approximate Solution	Exact Solution	Error
0.0	0.999999989	1.00000000	1.10E-08
0.1	0.953449454	0.953462589	1.31E-05
0.2	0.912840084	0.912870929	3.08E-05
0.3	0.877040782	0.877058019	1.72E-05
0.4	0.845161266	0.845154255	7.01E-06
0.5	0.81651405	0.816496581	1.75E-05
0.6	0.790576417	0.790569415	7.00E-06
0.7	0.766952398	0.766964989	1.26E-05
0.8	0.745334745	0.745355992	2.13E-05
0.9	0.725466911	0.72547625	9.34E-06
1.0	0.707105022	0.707106781	1.76E-06

Example 4.2

$y'(x) - x^2 y(x) = 0, \quad y(0) = 1, \quad y(x) = e^{\frac{x^3}{3}}, \quad 0 \le x \le 1$

The solution obtained are

For $\omega(x) = 1 + x$

$y_5(x) = 1 + 0.01167524x - 0.096134109x^2 + 0.647352371x^3 - 0.465907229x^4 + 0.29909976x^5$

For $\omega(x) = 1 - x$

$y_5(x) = 1 + 1.75253535 \times 10^{-3}x - 0.024896601x^2 + 0.457484632x^3 - 0.252209978x^4 + 0.21391392x^5$

For $\omega(x) = 1 - x^2$

$y_5(x) = 1 + 5.82006608 \times 10^{-3}x - 0.055171381x^2 + 0.53817526x^3 - 0.343315245x^4 - 0.250116478x^5$

The tau method also yields the solution:

$y_5(x)1 + 1.62187603 \times 10^{-3}x - 0.038287302x^2 + 0.521664293x^3 - 0.350374703x^4 - 0.26097774x^5$

Computational results are presented in Tables.

Table-4.2(a) : Numerical Results For Example 4.2, w(x) = 1 + x

X_k	Approximate Solution	Exact Solution	Error
0.0	1.000000000	1.00000000	0.0
0.1	1.000809936	1.000333389	4.77E-4
0.2	1.003018763	1.002670225	3.49E-4
0.3	1.00928198	1.009040622	2.41E-4
0.4	1.021854747	1.021562516	2.92E-4
0.5	1.042950805	1.042546905	4.04E-4
0.6	1.075101397	1.074655344	4.46E-4
0.7	1.121514189	1.121125771	3.88E-4
0.8	1.186432185	1.186095318	3.37E-4
0.9	1.275492651	1.275068624	4.24E-4
1.0	1.396086033	1.395612425	4.74E-4

Table-4.2(b) : Numerical Results For Example 4.2, $\omega(x) = 1 - x$

X_k	Approximate Solution	Exact Solution	Error
0.0	1.00000000	1.00000000	0.0
0.1	1.00036064	1.000333389	2.73E-05
0.2	1.002678637	1.002670225	8.41E-06
0.3	1.009110012	1.009040622	6.94E-05
0.4	1.021717678	1.021562516	1.55E-04
0.5	1.042728133	1.042546905	1.81E-04
0.6	1.074788159	1.074655344	1.33E-04
0.7	1.121221516	1.121125771	9.57E-05
0.8	1.186285642	1.186095318	1.90E-04
0.9	1.275428346	1.275068624	3.60E-04
1.0	1.395544508	1.395612425	6.79E-05

Table-4.2(c) : Numerical Results For Example 4.2, $\omega(x) = 1 - x$

X_k	Approximate Solution	Exact Solution	Error
0.0	1.00000000	1.00000000	0.0
0.1	1.000536638	1.000333389	2.03E-04
0.2	1.002793293	1.002670225	1.23E-04
0.3	1.009138257	1.009040622	9.76E-05
0.4	1.021716145	1.021562516	1.54E-04
0.5	1.042748032	1.042546905	2.01E-04
0.6	1.0748316	1.074655344	1.76E-04
0.7	1.12124127	1.121125771	1.16E-04
0.8	1.186228345	1.186095318	1.33E-04
0.9	1.275321152	1.275068624	2.53E-04
1.0	1.395625178	1.395612425	1.28E-05

Table-4.2(d) : Numerical Results For Example 4.2, Chebyshev

X_i	Approximate Solution	Exact Solution	Error
0.0	1.00000000	1.00000000	0.0
0.1	1.000268551	1.000333389	6.48E-05
0.2	1.002489111	1.002670225	1.81E-04
0.3	1.008921782	1.009040622	1.19E-04
0.4	1.021612117	1.021562516	4.96E-05
0.5	1.042704286	1.042546905	1.57E-04
0.6	1.074754254	1.074655344	9.89E-05
0.7	1.121042956	1.121125771	8.28E-05
0.8	1.185889464	1.186095318	2.06E-04
0.9	1.274964167	1.275068624	1.05E-04
1.0	1.395601938	1.395612425	1.05E-05

Remark

Results for the approximation of degrees n = 6 and 7 were similarly obtained but could not be displayed here due to space.

However, in chapter-5 a table which shows the maximum errors for all the cases we have considered is presented and discussed.

5. Conclusion

5.1 Summary of Results

We summarize the results of the preceding section by way of comparing the maximum errors in the various approximants of y(x) as obtained there, in the Table-5.1 below

Table-5.1 : Maximum Errors for problem 4.1-problem 4.2

Problems	4.1			4.2		
degree	5	6	7	5	6	7
$\omega(x) = 1 - x$	2.28E-04	6.19E-06	1.06E-06	3.64E-04	4.91E-05	1.53E-05
$\omega(x) = 1 + x$	6.70E-05	1.17E-05	2.02E-06	5.03E-03	7.06E-05	1.25E-05
$\omega(x) = 1 - x^2$	4.01E-05	6.95E-06	1.20E-06	2.64E-04	5.54E-05	6.78E-06
Chebyshev	3.08E-05	3.93E-06	6.64E-07	2.07E-04	2.96E-05	4.50E-06

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